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533. SOME INEQUALITIES RELATED TO A TRIANGLE*

Branko M. Milisavljević

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Editorial Committee

In this Note we shall prove some inequalities for a triangle. We shall use the same notatios as in [1].

Theorem 1. *In every triangle*

$$\frac{7Rr-2r^2}{Rr} \leqq \sum \frac{b+c}{a} \leqq \frac{2R^2+Rr+2r^2}{Rr},$$

with equality if and only if the triangle is equilateral.

Proof. Since

$$\sum \frac{b+c}{a} = \frac{s^2-2Rr+r^2}{2Rr},$$

in virtue of (see [1], **5.9**)

$$16Rr-5r^2 \leqq s^2 \leqq 4R^2+4Rr+3r^2, \tag{1}$$

where equality occurs if and only if the triangle is equilateral, we obtain the statement of Theorem 1.

REMARK. Since $\sum a(h_b+h_c) = 2F \sum \frac{b+c}{a}$, from Theorem 1, inequalities

$$\frac{2F(7Rr-2r^2)}{Rr} \leqq \sum a(h_c+h_b) \leqq \frac{2F(2R^2+Rr+2r^2)}{Rr},$$

follow, which are sharper than inequalities obtained in [2].

Theorem 2. *For every triangle, inequalities*

$$\frac{R^2+3Rr+2r^2}{2R^2+3Rr+2r^2} \leqq \sum \frac{s-a}{b+c} \leqq \frac{6R}{9R-2r}, \tag{2}$$

hold. The equality is valid if and only if the triangle is equilateral.

Proof. From the identity

$$\sum \frac{s-a}{b+c} = \frac{1}{2}\left(1+\frac{2r(3R+2r)}{s^2+r(2R+r)}\right),$$

and on the basis of (1), inequalities (2) follow.

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Theorem 3. *In every triangle*

$$\frac{2}{R} \leqq \sum \frac{h_a}{w_a^2} \leqq \frac{1}{r},$$

where equality holds if and only if the tiringle is equilateral.

Proof. Since

$$\frac{R+2r}{2Rr} \geqq \frac{2}{R} \quad \text{and} \quad \frac{R+2r}{2Rr} \leqq \frac{1}{r},$$

from identity

$$\sum \frac{h_a}{w_a^2} = \frac{R+2r}{2Rr}$$

the statement of the theorem follows.

Theorem 4. *For every triangle inequalities*

$$9r \leqq \sum \frac{w_a^2}{h_a} \leqq 4R + r \tag{3}$$

are valid.

Proof. Since $w_a \geqq h_a$, $w_b \geqq h_b$, $w_c \geqq h_c$ and $\sum h_a \geqq 9r$, the first inequality (3) directly follows.

On the basis of identities

$$\sum \frac{w_a^2}{h_a} = 8sR \sum \frac{s-a}{(b+c)^2} \quad \text{and} \quad \sum \frac{s-a}{bc} = \frac{4R+r}{2sR},$$

it follows one after the other

$$\sum \frac{w_a^2}{h_a} \leqq 8sR \sum \frac{s-a}{4bc} = 4R + r.$$

Thereby the second inequality in (3) is proved, too.

Since in all the above inequalities, equality holds if and only if the triangle is equilateral, it follows that in (3) equality holds if and only if the triangle is equilateral.

Theorem 5. *In every triangle*

$$aw_a + bw_b + cw_c \geqq 6sr.$$

Equality holds if and only if the triangle is equilateral.

Proof. On the basis of inequalities $w_a \geqq h_a$, $w_b \geqq h_b$, $w_c \geqq h_c$, where equality holds if and only if the triangle is equilateral, we get

$$\sum aw_a \geqq \sum ah_a = \sum a \frac{2F}{a} = 6sr,$$

whereby the theorem is proved.

Theorem 6. *For every triangle*

$$6sr \leqq \sum am_a \leqq \frac{\sqrt{3}}{2} \sum a^2. \tag{4}$$

Equality holds if and only if the triangle is equilateral.

Proof. The first inequality is proved in the same way as the statement of Theorem 5.

On the basis of CAUCHY-SCHWARZ inequality, we have

(5) $$\sum am_a \leqq \left(\sum a^2\right)^{1/2} \left(\sum m_a^2\right)^{1/2}.$$

Since $\sum m_a^2 = \frac{3}{4} \sum a^2$ from (5) the other inequality in (4) follows, whereby the theorem is proved.

COMMENTS BY R. R. JANIĆ

1° If the triangle is non-obtuse then (see [3])

(6) $$s^2 \geqq 2R^2 + 8Rr + 3r^2.$$

Using this inequality and $\sum \frac{b+c}{a} = \frac{s^2 - 2Rr + r^2}{2Rr}$, we obtain

$$\frac{R^2 + 3Rr + 2r^2}{Rr} \leqq \sum \frac{b+c}{a},$$

i. e., if the triangle is non-obtuse theo the following inequalities are valid

$$\frac{R^2 + 3Rr + 2r^2}{Rr} \leqq \sum \frac{b+c}{a} \leqq \frac{2R^2 + Rr + 2r^2}{Rr}$$

which are sharper than inequalities mentioned in Theorem 1.

2° In the same way we obtain

$$\sum \frac{s-a}{b+c} \leqq \frac{R^2 + 8Rr + 4r^2}{2R^2 + 10Rr + 4r^2}$$

which is sharper than the right inequality in (2).

3° Since (see [4])

$$\frac{m_a}{w_a} \geqq \frac{(b+c)^2}{4bc} \geqq 1,$$

we have

(7) $$aw_a \leqq am_a.$$

Similarly

(8) $$bw_b \leqq bm_b, \quad cw_c \leqq cm_c.$$

Adding (7) and (8) we get

$$\sum aw_a \leqq \sum am_a.$$

Therefore

$$6rs \leqq \sum aw_a \leqq \sum am_a \leqq \frac{\sqrt{3}}{2} \sum a^2.$$

The previous inequalities can be written in the symmetrical form, i.e.

$$6rs \leqq \sum aw_a \leqq \sum am_a \leqq 3Rs.$$

Let p_a, p_b, p_c be the distances of the circumcentre from the sides BC, CA, AB respectively. Then

$$m_a \leqq R + p_a, \quad m_b \leqq R + p_b, \quad m_c \leqq R + p_c,$$

which implies

$$am_a + bm_b + cm_c \leqq R(a+b+c) + ap_a + bp_b + cp_c.$$

Since

$$ap_a = R^2 \sin 2\alpha, \quad bp_b = R^2 \sin 2\beta, \quad cp_c = R^2 \sin 2\gamma$$

and (see [1] **2.4.** p. 18)

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma \leqq \sin\alpha + \sin\beta + \sin\gamma = \frac{a+b+c}{2R}$$

we obtain

$$am_a + bm_b + cm \leqq 2\,Rs + Rs = 3\,Rs.$$

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