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507. SEXTIC INEQUALITIES FOR THE SIDES OF A TRIANGLE*

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1. Many of the interesting inequalities connecting the sides, angles, radii etc. of a triangle, such as can be found in [**3**] and elsewhere, can be reduced by well known formulae to symmetric homogeneous polynomial inequalities in the sides a, b, c of the triangle. Symmetric homogeneous cubic inequalities have been completely investigated in [**1**] and [**5**], and quartic inequalities partially investigated in [**1**].

We can write $a=y+z$, $b=z+x$, $c=x+y$ where x, y, z are positive; conversely, if x, y, z are positive then $y+z$, $z+x$, $x+y$ are the sides of a triangle. It is convenient to work with x, y, z rather than with a, b, c.

A number of existing inequalities can be reduced to symmetric homogeneous sextic inequalities in x, y, z, *with equality when* $x=y=z$ *and without any terms involving* $\sum x^6$ *or* $\sum x^5(y+z)$. In this paper we shall find necessary and sufficient conditions for such inequalities to be satisfied for all positive x, y, z; we shall then use these conditions to prove some new and some old inequalities.

The term „triangle" will mean „triangle or degenerate triangle", „positive" will mean „positive or zero", „acute triangle" will mean „acute or right triangle" etc. Terms such as „strict triangle" will be used when we wish to exclude degenerate cases.

2. Let us write $J=\sum x^4(y^2+z^2)$, $K=\sum x^4yz=xyz\sum x^3$, $L=\sum y^3z^3$, $M=\sum(x^3y^2z+x^3yz^2)=xyz\sum x^2(y+z)$, $N=x^2y^2z^2$. We start with some simple inequalities; others can easily be obtained in a similar way, but they turn out to be positive linear combinations of those given below.

(1) $$\sum x^2yz(y-z)^2=M-6N\geqq 0$$

with equality if and only if $x=y=z$ or one of x, y, z equals zero.

(2) $$\sum(y-z)^2(z-x)^2(x-y)^2=J-2K-2L+2M-6N\geqq 0$$

with equality if and only if any two of x, y, z are equal.

The inequality

(3) $$\sum x^3-\sum x^2(y+z)+3xyz\geqq 0$$

is the special case $n=1$ of SCHUR's inequality $\sum x^n(x-y)(x-z)\geqq 0$ [**4**, p. 64], which is easily proved by writing

$$\sum x^n(x-y)(x-z)=x^n(x-y)^2+(x^n-y^n+z^n)(x-y)(y-z)+z^n(y-z)^2,$$

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and assuming without loss of generality that $x \geqq y \geqq z$. Equality occurs if and only if either $x=y=z$ or $x=0,\ y=z$ etc. (assuming $n \geqq 0$). The inequality (3) iz equivalent to COLINS's inequality [3, 1.6]; from it we obtain

$$(4) \qquad xyz\left(\sum x^3-\sum x^2(y+z)+3\,xyz\right)=K-M+3\,N\geqq 0$$

with equality if and only if $x=y=z$ or one of x, y, z equals zero.

Applying (3) to yz, zx, xy we obtain

$$(5) \qquad \sum(yz)^3-\sum(yz)^2(zx+xy)+3\,yz\cdot zx\cdot xy=L-M+3\,N\geqq 0$$

with equality if and only if $x=y=z$ or any two of x, y, z equal zero.

Let us write $J-2K-2L+2M-6N=P$, $K-M+3N=Q$, $L-M+3N=R$, $M-6N=S$; the inequalities (2), (4), (5) and (1) then become $P\geqq 0$, $Q\geqq 0$, $R\geqq 0$, $S\geqq 0$.

Any symmetric homogeneous sextic with no terms involving $\sum x^6$ or $\sum x^5(y+z)$ is a linear combination of J, K, L, M, N and can be written in the form

$$\alpha(J-2K-2L+2M-6N)+\beta(K-M+3N)+\gamma(L-M+3N)+\delta(M-6N)+\varepsilon N;$$

if we require this sextic to have value zero when $x=y=z$, we must have $\varepsilon=0$. The sextic can then be written in the form

$$f(x, y, z)=\alpha P+\beta Q+\gamma R+\delta S,$$

and we certainly have $f(x, y, z)\geqq 0$ for all positive x, y, z if $\alpha, \beta, \gamma, \delta\geqq 0$.

Suppose conversely that $f(x,y,z)\geqq 0$ for all positive x,y,z. Then $f(x, 1, 0)=x^2[\alpha(x-1)^2+\gamma x]$, so $x^{-4}f(x, 1, 0)\to\alpha$ as $x\to\infty$; hence $\alpha\geqq 0$. Also $x^{-4}f(x, 1, 1)\to\beta$ as $x\to\infty$; hence $\beta\geqq 0$. Also $f(0, 1, 1)=\gamma$; hence $\gamma\geqq 0$.

However, the following inequalities all show that δ need not be positive but can take the value $-\sqrt{\beta\gamma}$:

$$(6) \qquad \sum yz(x-y)^2(x-z)^2=Q+R-S\geqq 0;$$

$$(7) \qquad \sum(y-z)^2(x-\lambda y)^2(x-\lambda z)^2=(\lambda^4+\lambda^2+1)P+2(\lambda-1)^2[\lambda^2 Q+R-\lambda S]\geqq 0;$$

$$(8) \qquad \sum[(z-x)(y-\lambda z)(y-\lambda x)-(x-y)(z-\lambda x)(z-\lambda y)]^2$$
$$=(\lambda^4-\lambda^3-\lambda+1)P+3(\lambda-1)^2[\lambda^2 Q+R-\lambda S]\geqq 0.$$

Since $\beta, \gamma\geqq 0$, we can write

$$f(x, y, y)=y^2(x-y)^2\left[(x\sqrt{\beta}-y\sqrt{\gamma})^2+2(\delta+\sqrt{\beta\gamma})\,xy\right].$$

This expression is positive for all positive x, y if and only if $\delta\geqq-\sqrt{\beta\gamma}$; hence we have a necessary condition on δ for $f(x, y, z)$ to be always positive. We shall show that the condition is also sufficient by proving the following lemma.

Lemma. *If* $\beta, \gamma\geqq 0$ *then*

$$f_{\beta,\gamma}(x, y, z)=\beta Q+\gamma R-\sqrt{\beta\gamma}\,S\geqq 0$$

for all positive x, y, z. *If* $\gamma=0$, *equality occurs if and only if* $x=y=z$ *or one of* x, y, z *equals zero. If* $\gamma\neq 0$, *equality occurs if and only if* $x=y=z$ *or two of* x, y, z *equal zero or* $x:y:z=\sqrt{\gamma}:\sqrt{\beta}:\sqrt{\beta}$ *etc.*

Proof. (i) Suppose $\beta \leqq \gamma$. Assume without loss of generality that $x \geqq y \geqq z$. Then

$$4 f_{\beta,\gamma}(x, y, z) = yz(x-y)(x-z)\left(2x\sqrt{\beta} - y\sqrt{\gamma} - z\sqrt{\gamma}\right)^2$$
$$+ 4\left(\sqrt{\gamma} - \sqrt{\beta}\right)^2 (y-z)^2 x^2 yz + 4(\gamma-\beta)(y-z)^2(2x-y-z)xyz$$
$$+ \gamma(y-z)^2\left[4x^3(y+z) - 17x^2yz + 5xyz(y+z) - y^2z^2\right];$$

the expression in square brackets in the final term is positive since it can be written in the form

$$y(x-y)^3 + x(3y+4z)(x-y)^2 + y(x-y)(y-z)(9x+3y-5z) + 4y^2(y-z)^2.$$

(ii) Suppose $\beta > \gamma$. If $x=0$ or $y=0$ or $z=0$, the result is trivial. If not, then

$$f_{\beta,\gamma}(x, y, z) = x^4 y^4 z^4 f_{\gamma,\beta}(x^{-1}, y^{-1}, z^{-1}),$$

which is positive by (i) since $\gamma \leqq \beta$.

The values of x, y, z for which $f_{\beta,\gamma}(x, y, z) = 0$ are easily obtained from the above expression for $f_{\beta,\gamma}(x, y, z)$ as a sum of non-negative terms.

Let us write $\alpha P + \beta Q + \gamma R + \delta S = \Phi(\alpha, \beta, \gamma, \delta)$. Combining the above results we have the main theorem.

Theorem. *$\Phi(\alpha, \beta, \gamma, \delta) \geqq 0$ for all positive x, y, z if and only if $\alpha, \beta, \gamma \geqq 0$ and $\delta \geqq -\sqrt{\beta\gamma}$.*

The values of x, y, z for which equality occurs depend on $\alpha, \beta, \gamma, \delta$. In any particular case these values of x, y, z can be found by writing

$$\Phi(\alpha, \beta, \gamma, \delta) = \alpha P + f_{\beta,\gamma}(x, y, z) + \left(\delta + \sqrt{\beta\gamma}\right) S.$$

The rest of the paper is devoted to applications of this theorem. It is interesting to note that in most of the main applications either $\beta = 0$, or $\gamma = 0$, so that we only need the inequalities (1), (2), (4) and (5).

3. Consider first for what values of λ and μ the inequality

$$s^2 \leqq \lambda R^2 + \mu Rr + (27 - 4\lambda - 2\mu) r^2$$

is satisfied for all triangles; the coefficient of r^2 has been chosen to give equality when $a=b=c$, i.e., when $x=y=z$. Using the identities $4RF = abc = (y+z)(z+x)(x+y)$, $rF = F^2/s = xyz$, $s^2F^2 = xyz(x+y+z)^3$, where F is the area of the triangle, we see that the above inequality is equivalent (except when $F=0$) to

$$\Phi(\lambda,\ 4\lambda - 16,\ 4\lambda,\ 12\lambda + 4\mu - 64) \geqq 0.$$

We therefore require $\lambda \geqq 4$. Write $\lambda = 4/(1-\theta^2)$ where $0 \leqq \theta < 1$; then $4\lambda - 16 = 16\theta^2/(1-\theta^2)$. We must also have

$$12\lambda + 4\mu - 64 \geqq -\sqrt{4\lambda(4\lambda - 16)} = -16\theta/(1-\theta^2),$$

i.e., $\mu = 4(1 - \theta - 4\theta^2)/(1-\theta^2) + \varepsilon^2$, say.

If we put $\varepsilon = 0$ we obtain

(9) $$s^2 \leqq (1-\theta^2)^{-1}\left[4R^2 + 4(1 - \theta - 4\theta^2) Rr + (3 + 8\theta + 5\theta^2) r^2\right].$$

Equality occurs here if and only if $x=y=z$ or $x:y:z=1:\theta:\theta$ etc., i.e., when $a=b=c$ or $a:b:c=2\theta:1+\theta:1+\theta$ etc. We must ignore the equality that occurs in the sextic when $y=z=0$ etc., since $F=0$ in these cases and we must divide by F to get back to the original inequality; also R is not defined in these degenerate cases.

The inequalities (9), for all allowable values of θ, are all best possible, in the sense that, if we make μ any bigger for a given value of λ, we simply add a positive multiple of the positive expression $r(R-2r)$ to the right hand side.

For different values of θ, the right hand sides of (9) cannot be compared, since they are equal to s^2 for different values of x, y, z. Thus BLUNDON's inequality $s^2 \leqq 4R^2+4Rr+3r^2$ [3, 5.8, 5.9], given by $\theta=0$, is just one of a whole range of best possible inequalities.

The value $\theta=1/3$ gives

$$2s^2 \leqq 9R^2+2Rr+14r^2=9R^2+8Rr+2r^2-6r(R-2r),$$

which improves [3, 5.6], and $\theta=1/2$ gives

$$3s^2 \leqq (4R+r)^2-16r(R-2r),$$

which improves [3, 5.5]. We can eliminate the term in Rr in (9) by putting $\theta=(\sqrt{17}-1)/8$; this given

$$(23+\sqrt{17})s^2 \leqq 128R^2+(109+27\sqrt{17})r^2.$$

The inequality $2s \leqq 3R\sqrt{3}$ [3, 5.3] is given by $\lambda=27/4$, $\mu=0$.

The opposite inequality

$$s^2 \geqq \lambda R^2+\mu Rr+(27-4\lambda-2\mu)r^2$$

is equivalent (except when $F=0$) to

$$\Phi(-\lambda,\ 16-4\lambda,\ -4\lambda,\ 64-12\lambda-4\mu) \geqq 0,$$

which leads to the best possible inequalities

(10) $$s^2 \geqq (1-\omega^2)^{-1}[-4\omega^2R^2+4(4+\omega-\omega^2)Rr-(5+8\omega+3\omega^2)r^2] \quad (0 \leqq \omega < 1).$$

Equality occurs here if and only if $x=y=z$ or $x:y:z=\omega:1:1$ etc.

The value $\omega=0$ gives

$$s^2 \geqq 16Rr-5R^2 \quad [3,\ 5.8,\ 5.9],$$

and $\lambda=\mu=0$ gives

$$s^2 \geqq 27r^2 \quad [3,\ 5.11].$$

The inequality $s^2 \leqq [\xi R+(3\sqrt{3}-2\xi)r]^2$ can be investigated in a similar way; the best possible value for ξ is 2 [3, 5.4].

4. Similar inequalities are also given in Chapter 5 of [3], with s^2 replaced by $\sum a^2$, $\sum bc$ and $\sum(b-c)^2$. These and other inequalities can be obtained from (9) and (10) with the aid of the identity

(11) $$2\sum bc-\sum a^2=4(4R+r)r.$$

For instance, to investigate $\sum a^2$ we write

$$\sum a^2 = 2s^2 - 2(4R + r)r.$$

As a final exemple of the use of (9), take $\theta = (\sqrt{17} - 3)/6$. By subtracting a suitable multiple of (11) from (9) we obtain

$$(3\sqrt{17} - 5)\sum a^2 + (15 - 3\sqrt{17})\sum bc \leqq 72(R^2 + r^2).$$

5. In [**3**, 11.18] two inequalities due to OPPENHEIM are given for the angles of an acute triangle.

If f, g, h are the sides of an acute triangle, then $g^2 + h^2 \geqq f^2$ etc.; if we write $f^2 = a$, $g^2 = b$, $h^2 = c$, then a, b, c are the sides of a triangle, and hence we may write $a = y + z$, $b = z + x$, $c = x + y$, where $x, y, z \geqq 0$. Conversely, if $x, y, z \geqq 0$ then f, g, h are the sides of an acute triangle.

If ξ, η, ζ are the angles of the acute triangle with sides f, g, h, then $\cos^2\xi = (g^2 + h^2 - f^2)^2/4g^2h^2 = x^2/(z + x)(x + y)$, etc..

Let us consider for what values of λ, μ, ν the inequality

$$64\lambda \prod \cos^2\xi + 16\mu \sum \cos^2\eta \cos^2\zeta + 4\nu \sum \cos^2\xi - (\lambda + 3\mu + 3\nu) \geqq 0$$

is satisfied for all acute triangles. This is equivalent (except when two of x, y, z equal zero) to

$$\Phi(\nu - \lambda - 3\mu,\ 4\nu - 4\lambda - 12\mu,\ 4\nu - 4\lambda + 4\mu,\ 4\nu - 12\lambda - 4\mu) \geqq 0.$$

Necessary and sufficient conditions for this are

$$4\nu - 4\lambda - 12\mu = 4\theta^2 \text{ (say)}, \ \theta \geqq 0,$$

$$4\nu - 4\lambda + 4\mu = 4\omega^2 \text{ (say)}, \ \omega \geqq 0,$$

$$4\nu - 12\lambda - 4\mu = -4\theta\omega + 2\varepsilon^2 \text{ (say)}.$$

This gives $4\mu = \omega^2 - \theta^2$, $4\lambda = (\theta + \omega)^2 - \varepsilon^2$, $4\nu = 2\theta^2 + 2\theta\omega + 4\omega^2 - \varepsilon^2$. Thus the general inequality of this type may be written in the form

$$(12) \qquad \theta^2\left[16\prod \cos^2\xi - 4\sum \cos^2\eta \cos^2\zeta + 2\sum \cos^2\xi - 1\right]$$
$$+ 4\omega^2\left[4\prod \cos^2\xi + \sum \cos^2\eta \cos^2\zeta + \sum \cos^2\xi - 1\right]$$
$$+ (-2\theta\omega + \varepsilon^2)\left[-16\prod \cos^2\xi - \sum \cos^2\xi + 1\right] \geqq 0.$$

If we write (12) in the form

$$\Phi(\theta^2,\ 4\theta^2,\ 4\omega^2,\ -4\theta\omega + 2\varepsilon^2) \geqq 0,$$

we can easily find the values of x, y, z for which equality occurs.

OPPENHEIM's inequalities [**3**, 11.18] are given by $\theta = 1$, $\omega = 0$, $\varepsilon = 1$ and by $\theta = 1$, $\omega = 0$, $\varepsilon = \sqrt{2}$. KOOISTRA's inequality $\sum \sec^2\xi \geqq 12$ [**3**. 2.46] is given by $\theta = 0$, $\omega = 1$, $\varepsilon = 2$.

6. The calculations involved in § 5 suggest that we should also investigate for what values of λ, μ, ν the inequality

$$-8\lambda \prod \cos\alpha - 4\mu \sum \cos\beta \cos\gamma - 2\nu \sum \cos\alpha + (\lambda + 3\mu + 3\nu) \geqq 0$$

is true for *all* triangles. We have $\cos\alpha = (b^2 + c^2 - a^2)/2bc$ etc., and the inequality is equivalent to

$$\Phi(9\lambda + 7\mu + \nu,\ 4\lambda + 12\mu + 4\nu,\ 36\lambda + 28\mu + 4\nu,\ 12\lambda + 20\mu + 4\nu) \geqq 0.$$

Necessary and sufficient conditions for this are

$$4\lambda + 12\mu + 4\nu = 4\theta^2 \quad \text{(say)},\ \theta \geqq 0,$$
$$36\lambda + 28\mu + 4\nu = 4\omega^2 \quad \text{(say)},\ \omega \geqq 0,$$
$$12\lambda + 20\mu + 4\nu = -4\theta\omega + 4\varepsilon^2 \quad \text{(say)}.$$

This gives $4\lambda = \theta^2 + 2\theta\omega + \omega^2 - 2\varepsilon^2$, $4\mu = -3\theta^2 - 4\theta\omega - \omega^2 + 4\varepsilon^2$, $2\nu = 6\theta^2 + 5\theta\omega + \omega^2 - 5\varepsilon^2$. Thus the general inequality of this type may be written in the form

$$\begin{aligned}(13)\quad & \theta^2[-2\prod\cos\alpha + 3\sum\cos\beta\cos\gamma - 6\sum\cos\alpha + 7] \\ & + \omega^2[-2\prod\cos\alpha + \sum\cos\beta\cos\gamma - \sum\cos\alpha + 1] \\ & + (-\theta\omega + \varepsilon^2)[4\prod\cos\alpha - 4\sum\cos\beta\cos\gamma + 5\sum\cos\alpha - 5] \geqq 0.\end{aligned}$$

If we write (13) in the form

$$\Phi(\omega^2,\ 4\theta^2,\ 4\omega^2,\ -4\theta\omega + 4\varepsilon^2) \geqq 0,$$

we can easily find the values of x, y, z for which equality occurs.

BAGER's inequality $\sum\cos\beta\cos\gamma \geqq 6\prod\cos\alpha$ [**1**, (8)] is given by $\theta = 0$, $\omega = \sqrt{5}$, $\varepsilon = 1$.

7. Let h_a, h_b, h_c denote the altitudes of the triangle with sides a, b, c. The expression $\sum h_a^2/a^2$ takes the value zero for degenerate triangles in which one angle measures 180°, but a reasonable conjecture for acute triangles is

$$(14)\qquad \sum h_f^2/f^2 \geqq 9/4,$$

with equality for equilateral triangles and isosceles right-angled triangles.

Using first the substitution $f^2 = a$ etc. as in § 5, we find that this conjecture is equivalent to

$$(15)\qquad (\sum b^2c^2)(2\sum bc - \sum a^2) \geqq 9a^2b^2c^2,$$

or

$$4\sum x^5(y+z) - J + 2K - 6L - 2M + 6N \geqq 0,$$

i.e.

$$4\sum xy(x-y)^4 + \Phi(15, 32, 0, 0) \geqq 0,$$

which is certainly true. Equality occurs if and only if $x = y = z$ or $y = z$, $x = 0$ etc. or $y = z = 0$ etc.. This gives $f = g = h$ or $f:g:h = \sqrt{2}:1:1$ etc.; the cases $g = h$,

$f=0$ etc. must be excluded when we revert to the original inequality since f, g, h occur in the denominator.

8. The value of $\sum h_f/f$ for right-angled triangles has minimum $5/2$ and maximum infinity, whereas its value when $f=g=h$ is $3\sqrt{3}/2>5/2$. Hence we conjecture that

(16) $$h_f/f \geqq 5/2 \text{ for acute triangles.}$$

This conjecture is equivalent to

(17) $$\left(\sum b^2c^2\right)\left(2\sum bc-\sum a^2\right)+2\,abc\left(\sum a\right)\left(2\sum bc-\sum a^2\right)\geqq 25\,a^2b^2c^2.$$

Setting that we have already proved (15) it will be sufficient to prove

(18) $$2\,abc\left(\sum a\right)\left(2\sum bc-\sum a^2\right)\geqq 16\,a^2b^2c^2,$$

which, after division by $2\,abc$, reduces to

(19) $$-\sum a^3+\sum a^2(b+c)-2\,abc\geqq 0.$$

This is equivalent to $8\,xyz\geqq 0$. Thus we have equality in (18) if and only if $x=0$ etc. (i.e., $a=b+c$ etc.) or $a=0$ etc.. Since (17) is the sum of (15) and (18) we have equality in (17) if and only if $f:g:h=\sqrt{2}:1:1$ etc., i.e., if and only if the triangle is an isosceles right-angled triangle.

9. Let P be a point inside the triangle ABC, and let AP, BP, CP meet the opposite sides at D, E, F respectively. Using (14) and (16) to deal with acute triangles, we can easily prove that

(20) $$\sum AD^2/BC^2\geqq 2 \text{ and } \sum AD/BC\geqq 2.$$

Equality occurs only in the degenerate cases when A is the mid-point of BC etc.

10. The inequality (19) can be written

$$\left(\sum a\right)\left(2\sum bc-\sum a^2\right)\geqq 8\,abc,$$

or

(21) $$\left(\sum f^2\right)\left(2\sum g^2h^2-\sum f^4\right)\geqq 8\,f^2g^2h^2.$$

If R and F denote the circumradius and area of the acute triangle with sides f, g, h, (21) can be written

$$\left(\sum f^2\right)16\,F^2\geqq 8\,(16\,R^2F^2).$$

Hence we have the inequality

(22) $$\sum f^2\geqq 8\,R^2 \text{ for acute triangles,}$$

with equality only for right-angled triangles.

11. A reasonable conjecture for acute triangles is

(23) $$\sum f^4\leqq 32\,R^4.$$

Using the usual substitutions we find that this conjecture is equivalent (except when $F=0$) to

$$\Phi(0,\ 0,\ 1,\ 2)+10\,N\geqq 0,$$

which is certainly true. Equality occurs if and only if two of x, y, z equal zero, i.e., if and only if $f=0$, $g=h$ etc.. In these cases $F=0$, but we can easily check that equality still holds in (23).

If the triangle ABC has a strictly obtuse angle at A, and if C' is diametrically opposite to B on the circumcircle, then $BC<BC'$ and $AC<AC'$. Hence $BC^4+CA^4+AB^4<BC'^4+C'A^4+AB^4\leqq 32\,R^4$ by (23), since ABC' is a right-angled triangle. Hence we have

(24) $$\sum a^4\leqq 32\,R^4$$

for all triangles.

Finally we shall prove

(25) $$\sum f^4\geqq 24\,R^4 \text{ for acute triangles.}$$

This is equivalent to

$$\Phi(1,\ 4,\ 0,\ 4)+24\,N\geqq 0.$$

Equality occurs if and only if $x=0$, $y=z$ etc., i.e., only for isosceles right-angled triangles.

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