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503. ADDITIONS TO KAMKE'S TREATISE, V: A REMARK ON THE GENERALISED EMDEN'S EQUATION*

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In his detailed paper [**1**] L. M. BERKOVIČ considered among other things, the following two generalised forms of EMDEN's differential equation

$$y''+f(x)\,y'+g(x)\,y^r=0 \qquad (r\in \mathbf{R}) \tag{1}$$

and

$$y''+f(x)\,y'+g(x)\,e^y=0, \tag{2}$$

and proved that if $g(x)$ has the form

$$g(x)=Ke^{-2\int f(x)\,\mathrm{d}x} \qquad (K=\text{const}) \tag{3}$$

then both equations (1) and (2) are integrable by quadratures.

Condition (3), which implies integrability of equations (1) and (2), was also obtained in numerous papers of I. BANDIĆ which are quoted in [**1**].

The object of this note is to give a simple proof of the same result for a differential equation which includes equations (1) and (2) as special cases. Namely, we prove the following

Theorem. *Differential equation*

$$y''+f(x)\,y'+g(x)\,h(y)=0 \tag{4}$$

is integrable by quadratures if $g(x)=Ke^{-2\int f(x)\,\mathrm{d}x}$.

Proof. We first solve the equation

$$y''+f(x)\,y'=0$$

to obtain

$$y'=Ce^{-\int f(x)\,dx}, \tag{5}$$

where C is an arbitrary constant. Suppose that C is a differentiable function of y. Then, differentiating (5) we find

$$y''=C'(y)\,C(y)\,e^{-2\int f(x)\,\mathrm{d}x}-C(y)f(x)\,e^{-\int f(x)\,\mathrm{d}x}. \tag{6}$$

Substituting (5) and (6) into (4) we obtain

$$C'(y)\,C(y)\,e^{-2\int f(x)\,\mathrm{d}x}+g(x)\,h(y)=0. \tag{7}$$

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Hence, if the condition (3) is fulfilled, (7) becomes

$$C'(y)\,C(y)+Kh(y)=0,$$

which implies

$$C(y)=\left(A-2K\int h(y)\,\mathrm{d}y\right)^{1/2}, \tag{8}$$

where A is an arbitrary constant.

Finally, substituting (8) into (5) we get

$$y'=\left(A-2K\int h(y)\,\mathrm{d}y\right)^{1/2} e^{-\int f(x)\,\mathrm{d}x},$$

whence we obtain the general solution of (4) in the form

$$\int\left(A-2K\int h(y)\,\mathrm{d}y\right)^{-1/2}\mathrm{d}y=\int e^{-\int f(x)\,dx}\,\mathrm{d}x+B,$$

where A and B are arbitrary constants.

Example. The Emden-Fowler equation (see **[2]**, equation 6.74)

$$xy''+2y'+ax^{\nu}y^{n}=0 \qquad (a>0)$$

is integrable by quadratures if $\nu=-3$. Its general solution is defined by

$$\int\left(A-\frac{2a}{n+1}\,y^{n+1}\right)^{-1/2}\mathrm{d}y=B-\frac{2}{x},$$

where A and B are arbitrary constants.

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Added in proof. The result of this note is proved, by a different method, in paper [3].

REFERENCES

1. Л. М. Беркович: *Об уравнении Фаулера-Эмдена и некоторых его обобщениях.* These Publications № **381**—№ **409** (1972), 51—62.
2. Э. Камке: *Справочник по обыкновенным дифференциальным уравнениям.* Москва 1971.
3. H. Karanikolov: *On a generalization of a generalized Emden equation.* Math. Balkanica **4** (1974), 317—318.