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502. ON COEFFICIENTS OF THE GAUSS — ENCKE FORMULA*

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The survey of the known results on coeficients K_{2n} of the Gauss — Encke formula

$$\frac{1}{h}\int\limits_{x}^{x+ph} f(x)\,\mathrm{d}x \sim \sum_{k=1}^{p} f\left(x+\left(k-\frac{1}{2}\right)h\right)+\sum_{n=1}^{+\infty} K_{2n}\left(\delta^{2n-1}f(x+ph)-\delta^{2n-1}f(x)\right)$$

is given in the present paper; the known asymptotic formula for K_{2n} is improved; an algorithm for quick calculation of these coefficients with the corresponding FORTRAN program is proposed, and a table of coeficients K_{2n} with 25 decimals is included.

Coefficients K_{2n} have the values

$$K_2=\frac{1}{24},\quad K_4=-\frac{17}{5760},\quad K_6=\frac{367}{967680},\quad K_8=-\frac{27859}{464486400},$$

$$K_{10}=\frac{1295803}{122624409600},\quad K_{12}=-\frac{5329242827}{2678117105664000},\ \ldots$$

T. OPPOLZER [**9**], p. 545, has exactly calculated K_{2n} for $n=1\,(1)\,10$ while H. E. SALZER [**11**], p. 217, has calculated K_{2n} for $n=11\,(1)\,25$ with 18 decimals.

In [**3**], p. 114, the connection of K_{2n} with BERNOULLI's polynomials

$$K_{2n}=\frac{1}{(2n)!\,(2n-1)}\,B_{2n}^{(2n-1)}\left(n-\frac{1}{2}\right)=-\frac{1}{(2n)!}\,B_{2n}^{(2n)}\left(n-\frac{1}{2}\right),$$

is given, while on p. 109 the values of $B_p^{(n)}\left(\frac{n}{2}\right)$ for $n=-6\,(1)\,6$ and $p=0\,(2)\,6$ are given. The generating function of BERNOULLI's polynomial $B_p^{(n)}(x)$ is

$$\frac{t^n e^{xt}}{(e^t-1)^n}=\sum_{p=0}^{+\infty} B_p^{(n)}(x)\,\frac{t^p}{p!},$$

see [**3**], p. 69.

In [**13**], pp. 1—15, and [**14**], pp. 107—109 and 190—191, J. F. STEFFENSEN has given the following formula

$$K_{2n}=\frac{1}{(2n)!}\sum_{k=1}^{n}\frac{4^{-k}}{(2k+1)!}\left(\frac{\mathrm{d}^{2k}}{\mathrm{d}x^{2k}}\prod_{j=0}^{k-1}(x^2-j^2)\right)_{x=0},$$

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(1) $$K_{2n} \sim \frac{(-1)^{n+1}}{2^{2n-1}\, n^{3/2}\, \pi^{5/2}} \qquad (n \to +\infty),$$

(2) $$K_{2n} = \frac{1^2}{3!\, 2^2} K_{2n-2} - \frac{(1\cdot 3)^2}{5!\, 2^4} K_{2n-4} + \cdots + (-1)^n \frac{(1\cdot 3\cdots(2n-3))^2}{(2n-1)!\, 2^{2n-2}} K_2 + (-1)^{n+1} \frac{(1\cdot 3\cdots(2n-1))^2}{(2n+1)!\, 2^{2n}}.$$

D. K. SEN [**10**] has obtained the result

(3) $$\sum_{n=1}^{+\infty} |K_{2n}| = 1 - \frac{3}{\pi}.$$

By means of STIRLING's interpolation formula

$$y = y_0 + \frac{1}{1!} u \mu\delta\, y_0 + \frac{1}{2!} u^2 \delta^2 y_0 + \frac{1}{3!} u(u^2 - 1^2) \mu\delta^3 y_0 + \cdots + \frac{1}{(2k-1)!} u (u^2 - 1^2)(u^2 - 2^2) \cdots \left(u^2 - (k-1)^2\right) \mu\delta^{2k-1} y_0 + \frac{1}{(2k)!} u^2 (u^2 - 1^2)(u^2 - 2^2) \cdots \left(u^2 - (k-1)^2\right) \delta^{2k} y_0 + \cdots,$$

$$\mu\delta^{2k-1} y_0 = \frac{1}{2} \left(\delta^{2k-1} y_{\frac{1}{2}} + \delta^{2k-1} y_{-\frac{1}{2}}\right)$$

the result

$$\frac{1}{h} \int_{x_0+\frac{h}{2}}^{x_p+\frac{h}{2}} f(x)\, dx = \sum_{n=1}^{m-1} K_{2n} \left[\delta^{2n-1} f\left(x_p + \frac{h}{2}\right) - \delta^{2n-1} f\left(x_0 + \frac{h}{2}\right)\right] + \sum_{k=1}^{p} f(x_k) - h^{2m} K_{2m} f^{(2m)}(\xi) \quad (x_{1-k} < \xi < x_{n+k}),$$

has been proved in [**6**], pp. 77 and 130, where

(4) $$K_{2n} = \frac{1}{(2n)!} \int_{-1/2}^{1/2} t^2 (t^2 - 1^2) \cdots \left(t^2 - (n-1)^2\right) dt.$$

(Compare [**1**], p. 269, see [**4**], Vol. 2, p. 98.)

V. I. KRYLOV and L. T. ŠUL'GINA have proved formula (4) by means of the second EULER—MACLAURIN formula

$$\frac{1}{h} \int_a^b f(x)\, dx = \sum_{p=1}^{n} f\left(a + \frac{2p-1}{2} h\right) + \sum_{k=1}^{m-1} \frac{h^{2k} (1 - 2^{1-2k}) B_{2k}}{(2k)!} \left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right] + \frac{h^{2m-1}}{(2m)!} (1 - 2^{1-2m})\, B_{2m} f^{(2m)}(\xi) \quad (a < \xi < b)$$

and by GAUSS' formulas for the derivatives of functions expressed in terms of central differences

$$f'(x)=\frac{1}{h}\left(\delta^1-\frac{1}{6}\delta^3+\frac{1}{30}\delta^5-\frac{1}{140}\delta^7+-\cdots\right)f(x),$$

$$f'''(x)=\frac{1}{h^3}\left(\delta^3-\frac{1}{4}\delta^5+\frac{7}{120}\delta^7-+\cdots\right)f(x),$$

$$f^{(V)}(x)=\frac{1}{h^5}\left(\delta^5-\frac{1}{3}\delta^7+-\cdots\right)f(x),$$

$$f^{(VII)}(x)=\frac{1}{h^7}\left(\delta^7-+\cdots\right)f(x);$$

(see [6], pp. 62—63, 112, and [2], p. 75).

Formula (4) is known in the literature as one of GAUSS' formula. In [7], p. 189, it is called the GAUSS—ENCKE formula.

According to (1) the absolute values for K_{2n} tend quickly to zero, so that, due to the manner of storing the real numbers in digital computer, we can calculate only a comparatively small number of coefficients K_{2n}.

That is why we introduce the substitution

(6) $$G_n=(-1)^{n+1}2^{2n}K_{2n},$$

so that

$$G_1=\frac{1}{6},\quad G_2=\frac{17}{360},\quad G_3=\frac{367}{15120},\quad G_4=\frac{27859}{1814400},\quad G_5=\frac{1295803}{119750400},\ \ldots$$

From (4) and (6) it follows

(7) $$G_n=(-1)^{n+1}\frac{2^{2n}}{n}\int_0^{1/2} t\binom{t+n-1}{2n-1}\,\mathrm{d}t,$$

while from (1) and (6) we get

(8) $$G_n\sim 2\pi^{-5/2}n^{-3/2}.$$

On the basis of formulas (2) and (6), and the relationship between the gamma function and the factorial $\Gamma(m+1)=m!$ and LEGENDRE's duplication formula

$$\frac{2^{z-1}}{\sqrt{\pi}}\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right)=\Gamma(z)$$

it follows that

(9) $$\sum_{k=1}^{n}\frac{\Gamma\left(n-k+\frac{1}{2}\right)}{(2n-2k+1)\Gamma(n-k+1)}G_k=\frac{\Gamma\left(n+\frac{1}{2}\right)}{(2n+1)\Gamma(n+1)},$$

or more generally

(10) $$\sum_{k=1}^{n}\frac{a_{n-k}}{2n-2k+1}G_k=\frac{a_n}{2n+1},\quad a_m=\frac{2m-1}{2m}a_{m-1}.$$

Taking $a_0 = 1$, from (9) and (10) it follows that

$$a_m = \frac{\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(m+1)},$$

so that the coefficients a_m have the values

$$a_1 = \frac{1}{2}, \quad a_2 = \frac{3}{8}, \quad a_3 = \frac{5}{16}, \quad a_4 = \frac{35}{128}, \quad a_5 = \frac{63}{256}, \ldots$$

Using BOYD's inequality

$$\left(m+\frac{1}{4}+\frac{1}{32m+32}\right)^{1/2} < \frac{\Gamma(m+1)}{\Gamma\left(m+\frac{1}{2}\right)} < \frac{m+\frac{1}{2}}{\left(m+\frac{3}{4}+\frac{1}{32m+48}\right)^{1/2}}, \tag{11}$$

(see for example [8], p. 281), we get the asymptotic relation

$$a_m \sim \frac{1}{\sqrt{\pi}}\left(m+\frac{1}{4}+\frac{1}{32m}\right)^{-1/2}.$$

For the calculation of coefficient G_n on a digital computer, the formula (10) is the most suitable, i.e.,

$$G_n = -\sum_{k=1}^{n} \frac{a_k}{2k+1} G_{n-k} \quad \left(G_0 = -1,\ a_1 = \frac{1}{2},\ a_k = \frac{2k-1}{2k} a_{k-1}\right). \tag{12}$$

However, in the calculation of G_n by (12), all previous coefficients $G_1, G_2, \ldots, G_{n-1}$, take part, so that the calculation time of the coefficient G_n is proportional to n and that of all coefficients $G_1, G_2, \ldots, G_n$ is proportional to n^2. Therefore, the algorithm based on (12) is usable only for small values of the index n.

We propose an algorithm whereby the calculation time of all n coefficients $G_1, G_2, \ldots, G_n$ is proportional to n. Integrating (8) we get

$$G_n = \sum_{k=1}^{n} \frac{w(k,n)}{2k+1}, \tag{13}$$

where

$$w(k,1) = \frac{1}{2}\delta_{k,1}, \quad w(k,n+1) = \frac{(\delta_{k,1}-1)\,w(k-1,n)-(6n+2)\,w(k,n)}{(2n+1)(2n+2)} + w(k,n). \tag{14}$$

As $|w(k,n)|$ decreases quickly with increasing k, it is sufficient to take

$$G_n \approx \sum_{k=1}^{L} \frac{w(k,n)}{2k+1} \tag{15}$$

instead of (13). The number of summands L and the number of exact decimal digits D are related by inequality $D \geqq 2L-6$.

```
      SUBROUTINE COEF(M,G)
      DIMENSION G(1),W(16)
      DATA L/16/
      DO 1 K=2,L
    1 W(K)=0
      W(1)=2
      W(1)=1/W(1)
      DO 5 N=1,M
      Q=0
      K=L
    2 Q=W(K)/(K+K+1)+Q
      K=K-1
      IF(K) 3,3,2
    3 G(N)=Q
      Q=N+N
      Q=Q*Q
      P=N+N+1
      P=P*P+P
      K=L
    4 W(K)=(W(K)*Q-W(K-1))/P
      K=K-1
      IF(K-1) 5,5,4
    5 W(1)=W(1)*Q/P
      RETURN
      END
```

Fig. 1

n	K_{2n}
1	0.04166 66666 66666 66666 66667
2	−0.00295 13888 88888 88888 88889
3	0.00037 92576 05820 10582 01058
4	−0.00005 99780 74707 89241 62257
5	0.00001 05672 51693 41814 30709
6	−0.00000 19899 21507 06520 06158
7	0.00000 03920 48718 88204 69138
8	−0.00000 00798 10091 39050 70251
9	0.00000 00166 55098 32389 98268
10	−0.00000 00035 43916 01596 84882
11	0.00000 00007 65988 01282 74155
12	−0.00000 00001 67709 06092 05260
13	0.00000 00000 37117 25157 79131
14	−0.00000 00000 08290 38235 52219
15	0.00000 00000 01866 36137 17356
16	−0.00000 00000 00423 04998 11656
17	0.00000 00000 00096 47109 69714
18	−0.00000 00000 00022 11620 26151
19	0.00000 00000 00005 09424 03677
20	−0.00000 00000 00001 17839 47462
21	0.00000 00000 00000 27362 96247
22	−0.00000 00000 00000 06375 88426
23	0.00000 00000 00000 01490 34823
24	−0.00000 00000 00000 00349 37153
25	0.00000 00000 00000 00082 11767
26	−0.00000 00000 00000 00019 34837
27	0.00000 00000 00000 00004 56909
28	−0.00000 00000 00000 00001 08124
29	0.00000 00000 00000 00000 25636
30	−0.00000 00000 00000 00000 06089
31	0.00000 00000 00000 00000 01449
32	−0.00000 00000 00000 00000 00345
33	0.00000 00000 00000 00000 00082
34	−0.00000 00000 00000 00000 00020
35	0.00000 00000 00000 00000 00005
36	−0.00000 00000 00000 00000 00001

Table 1

Fig. 1 shows the COEF program which calculates the coefficients for $N=1, \dots, M$. The auxiliary vector W has the dimension L. The program adopts $L=16$, which means that it enables the calculation of coefficients to at least 26 accurate decimal digits. Naturally, it is assumed that the computer can operate with such accuracy. By means of formula (6) and COEF program, the coefficients K_{2n} for $n=1\,(1)\,36$ have been calculated, and tabulated in Table 1 to 25 decimals. The values of Table 1 are verified by SEN's formula (3).

From (13) it follows that

$$w(1, n) = \frac{\sqrt{\pi}\,\Gamma(n)}{4n\,\Gamma\left(n+\frac{1}{2}\right)}$$

wherefrom, on the basis of (11) we get

$$\frac{w(1,n)}{3} \sim \frac{\sqrt{\pi}/12}{n^{3/2}} \sim \frac{\pi^3}{24} G_n \quad (n \to +\infty).$$

We can improve formula (7). Starting from

$$\operatorname{Log}\Gamma(z+a) = \left(z+a-\frac{1}{2}\right)\log z - z + \log\sqrt{2\pi}$$
$$- A\log(1+e^{2\pi Bz}) + \sum_{k=2}^{m} \frac{(-1)^k B_k(a)}{k(k-1)\,z^{k-1}} + O(z^{-m}),$$

$$A=1\,(x<0),\quad A=\frac{1}{2}\ (x=0),\quad A=0\ (x>0),\quad B=i\ (y\geqq 0),\quad B=-i\ (y<0),$$

(see [12], p. 73) for $m=2$ and $n\to+\infty$ from (8) it follows that

$$G_n\sim\frac{2\pi^{-1/2}}{k^{3/2}}\left(\left(1+\frac{1}{8k}\right)\frac{1}{\pi^2}+\frac{1}{k\pi^4}\int_0^{\pi/2}t^3\sin t\,dt\right),$$

i.e.

$$G_n\sim\frac{2\pi^{-5/2}n^{-3/2}}{1-\left(\frac{7}{8}-\frac{6}{\pi^2}\right)\frac{1}{n}}\qquad(n\to+\infty).$$

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