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498. THE CENTROID METHOD IN INEQUALITIES*

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1. INTRODUCTION

The concept of the centroid, introduced most likely by ARCHIMEDES, can be applied in solving various mathematical problems. We mention, for example, the papers of C. F. GAUSS [2] and L. FEJÉR [9]. In this paper we shall give a chronological account of the use of centroid in developing inequalities, pointing to some priorities which are neglected in the literature. Besides, using the centroid method, we shall prove some general inequalities which present complementary inequalities for the JENSEN inequality for convex functions. These inequalities contain several results which were earlier derived in different ways. Some of our inequalities are sharper than the known inequalities.

In mathematical literature (including the very authors who used the centroid method) there exists a difference in opinion whether inequalities obtained by the centroid method are really proved, or is the centroid method only a sort of a geometric interpretation, i. e. a method which may suggest an inequality which still remains to be proved analytically. The book [23] of M. B. BALK brings an argumented explanation why the centroid method can be taken as a method of proof just like any other method used in Mathematics for proving various theorems.

2. HISTORY

2.1. Jensen's inequality. If f is a convex function on $[a, b]$, then for any points $x_1, \ldots, x_n \in [a, b]$ and any positive numbers $p_1, \ldots, p_n$ we have

$$f\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i}\right) \leqq \frac{\sum_{i=1}^{n} p_i f(x_i)}{\sum_{i=1}^{n} p_i}. \tag{2.1.1}$$

This inequality is known in the literature as JENSEN's inequality. It was proved under the assumption that f is a J-convex function, i. e. a function such that

$$f\left(\frac{x+y}{2}\right) \leqq \frac{f(x)+f(y)}{2} \qquad (x, y \in [a, b])$$

and that $p_1, \ldots, p_n$ are positive numbers by J. L. W. V. JENSEN in 1905 [7, 8]. He applied the famous inductive method used by CAUCHY [1] in the proof of

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the arithmetic-geometric mean inequality. However, inequality (2.1.1) appears, under different assumptions, much earlier. JENSEN himself mentioned in the appendix to his paper that O. HÖLDER [5] proved inequality (2.1.1) in 1889, supposing that f is a twice differentiable function on $[a, b]$ such that $f''(x) \geqq 0$ on that interval. This supposition is in the case of twice differentiable functions equivalent with the supposition that f is convex. The above inequality was proved, after HÖLDER, using the same assumptions by R. HENDERSON [6] in 1895. However, as far back as 1875 a particular case of the above inequality, the case when $p_1 = \cdots = p_n$ was proved by J. GROLOUS [3] by an application of the centroid method. This is, as far as we could find, the first inequality for convex functions to appear in the mathematical literature. J. GROLOUS also introduced the assumption that $f''(x) > 0$, but it can be seen from the text itself that it is enough to assume that f is a convex function, in the geometric sense (see, for instance, D. S. MITRINOVIĆ [30, p. 15] and N. BOURBAKI [21]).

Inequality (2.1.1) is certainly the inequality which was proved by the centroid method the largest number of times. Among others, we mention that it is proved by that method in the papers of H. LOB [10], M. N. NARASIMHA IYENGAR [13], M. TOMIĆ [19], A. N. LOWAN [20], A. BARTON [29] and in the book [23] of M. B. BALK. None of these authors mentions earlier papers, while M. B. BALK mentions only paper [20] of A. N. LOWAN.

We shall give a proof of (2.1.1) by the centroid method in Section 3.

2.2. Čebyšev's inequality. Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two real sequences such that

$$a_1 \leqq \cdots \leqq a_n \quad \text{and} \quad b_1 \leqq \cdots \leqq b_n \quad \text{or} \quad a_1 \geqq \cdots \geqq a_n \quad \text{and} \quad b_1 \geqq \cdots \geqq b_n.$$

Then the following inequality is valid

$$\left(\frac{1}{n} \sum_{i=0}^{n} a_i\right)\left(\frac{1}{n} \sum_{i=0}^{n} b_i\right) \leqq \frac{1}{n} \sum_{i=0}^{n} a_i b_i. \tag{2.2.1}$$

This inequality was proved by the centroid method by É. PICARD in 1881. PICARD's proof is given in [4] by CH. HERMITE. The same proof was reproduced in paper [32][1].

In the book [23] of M. B. BALK there are two proofs of inequality (2.2.1) (without any bibliographical information). ST. I. GHEORGHITZA [31] also gave a version of the proof of (2.2.1) by the centroid method. We shall quote the improved version of the GHEORGHITZA proof, supplied by Professor P. R. BEESACK on the occasion of perusal of the first version of this paper.

Let $P_1, \ldots, P_n$ be points on the OX axis with abscissae $b_1, \ldots, b_n$ $(b_i \geqq 0,\ i = 1, \ldots, n)$ respectively. Fix to the point P_k a mass $a_k \geqq 0\ (k = 1, \ldots, n)$. The centroid of the system of masses a_k at the points with abscissae b_k $(1 \leqq k \leqq n)$ is given by

$$x_T = \frac{a_1 b_1 + \cdots + a_n b_n}{a_1 + \cdots + a_n}.$$

[1] Notice that in this paper, on page 5, 7 th row from below, the following sentence is omitted by mistake: „... vont en s'éloignant de l'origine, car...".

Now, consider the system of n equal masses $a=\frac{1}{n}\sum_{k=1}^{n} a_k$ (having the same total mass as the preceding) situated at the same points $b_1, \ldots, b_n$. The new

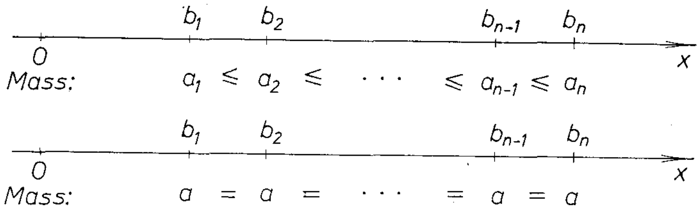

Fig. 1

mass distribution has clearly shifted to the left and so its centroid $x_{T'}$ has $x_{T'} \leqq x_T$, that is

$$x_{T'}=\frac{\sum_{k=1}^{n} ab_k}{\sum_{k=1}^{n} a}=\frac{1}{n}\sum_{k=1}^{n} b_k \leqq \frac{\sum_{k=1}^{n} a_k b_k}{\sum_{k=1}^{n} a_k}=x_T,$$

which implies (2.2.1).

Now, for the general case, there exist $\alpha>0$, $\beta>0$ such that $A_k \equiv a_k+\alpha \geqq 0$, $B_k \equiv b_k+\beta \geqq 0$ $(k=1, \ldots, n)$. Moreover $\{A_k\}$, $\{B_k\}$ are also either both increasing or both decreasing. Hence

$$\frac{1}{n}\sum_{k=1}^{n}(a_k+\alpha)\frac{1}{n}\sum_{k=1}^{n}(b_k+\beta) \leqq \frac{1}{n}\sum_{k=1}^{n}(a_k+\alpha)(b_k+\beta),$$

which reduces to (2.2.1).

2.3. The inequality on rearrangements. By the same centroid method used in **2.2.** one can give a proof of the basic inequality on rearrangements of finite sequences $a=(a_1, \ldots, a_n)$, $b=(b_1, \ldots, b_n)$. Let $\bar{a}=(\bar{a}_1, \ldots, \bar{a}_n)$ be the rearrangement of a in increasing order, and $\underline{a}=(\underline{a}_1, \ldots, \underline{a}_n)$ be the rearrangement of a in decreasing order, and define $\bar{b}$ and $\underline{b}$ analogously. Then

$$\sum_{i=1}^{n} \bar{a}_i \underline{b}_i \leqq \sum_{i=1}^{n} a_i b_i \leqq \sum_{i=1}^{n} \bar{a}_i \bar{b}_i. \tag{2.3.1}$$

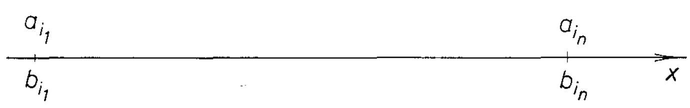

Fig. 2

To apply the centroid method, assume first that all $a_i \geqq 0$ and interpret the a_i as the mass of a particle at abscissa $b_i (1 \leqq i \leqq n)$. Then (with obvious notation) the centroids of the three systems in question satisfy

$$\underline{x} = \frac{\sum_{i=1}^{n} \overline{a_i}\, \underline{b_i}}{\sum_{i=1}^{n} \underline{b_i}} \leqq x = \frac{\sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} b_i} \leqq \overline{x} = \frac{\sum_{i=1}^{n} \overline{a_i}\, \overline{b_i}}{\sum_{i=1}^{n} \overline{b_i}},$$

which reduces to (2.3.1). If some $a_i < 0$, apply (2.3.1) to $A_i = a_i + \alpha > 0$. This proof is given in [**14**], p. 262.

2.4. Some complementary inequalities. Let $a_i, b_i (i = 1, \ldots, n)$ be real numbers such that

$$0 < m_1 \leqq a_i \leqq M_1, \quad 0 < m_2 \leqq b_i \leqq M_2 \qquad (i = 1, \ldots, n). \tag{2.4.1}$$

Then the following inequalities are valid:

1° Inequality of PÓLYA-SZEGÖ [**11**] or [**30**, p. 60]:

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leqq \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2} \left(\sum_{i=1}^{n} a_i b_i \right)^2; \tag{2.4.2}$$

2° Inequality of KANTOROVIČ [**18**] or [**30**, p. 60]:

$$\sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} \frac{p_i}{a_i} \leqq \frac{(M_1 + m_1)^2}{4 m_1 M_1} \quad (p_i \geqq 0;\ i = 1, \ldots, n); \tag{2.4.3}$$

3° Inequality of GREUB-RHEINBOLDT [**22**] or [**30**, p. 60]:

$$\sum_{i=1}^{n} p_i a_i^2 \sum_{i=1}^{n} p_i b_i^2 \leqq \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2} \left(\sum_{i=1}^{n} p_i a_i b_i \right)^2; \tag{2.4.4}$$

4° Inequality of SPECHT [**24**], [**25**] or [**30**, p. 79]:

$$\frac{M_n^{[s]}(a; p)}{M_n^{[t]}(a; p)} \leqq \Gamma_{s,t}, \tag{2.4.5}$$

where $t < s$, $C = \dfrac{M_1}{m_1}$, $M_n^{[s]}(a; p)$ is the weighted mean of order s and

$$\Gamma_{s,t} = \left(\frac{t (C^s - C^t)}{(s-t)(C^t - 1)} \right)^{\frac{1}{s}} \left(\frac{s (C^t - C^s)}{(t-s)(C^s - 1)} \right)^{-\frac{1}{t}} \qquad (st \neq 0),$$

$$\Gamma_{s,0} = \left(\frac{C^{s/(C^s-1)}}{e \log (C^{s/(C^s-1)})} \right)^{\frac{1}{s}}, \tag{2.4.6}$$

$$\Gamma_{0,t} = \left(\frac{(C^{t/(C^t-1)})}{e \log (C^{t/(C^t-1)})} \right)^{-\frac{1}{t}};$$

5° Inequality of GHEORGHIU [**12**]:

$$(2.4.7)\qquad \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}} \leqq C\left(\sum_{i=1}^{n} a_i b_i\right) \qquad \left(\frac{1}{p}+\frac{1}{q}=1;\ p>1\right),$$

where

$$(2.4.8)\quad C=\frac{M_1^p M_2^q - m_1^p m_2^q}{(pm_2 M_2(M_1 M_2^{q-1}-m_1 m_2^{q-1}))^{1/p}(qm_1 M_1(M_2 M_1^{p-1}-m_2 m_1^{p-1}))^{1/q}}.$$

For the history of these inequalities consult [**30**]. We shall give here only the history of the centroid method applied to the mentioned inequalities. We notice that we did not find inequality (2.4.7) quoted in the literature. Some thirty years later J. B. DIAZ, A. J. GOLDMAN and F. T. METCALF [**27**] obtained the integral analogue of (2.4.7) (which can be, in fact, obtained directly from (2.4.7)).

In the mentioned article [**12**] Ş. A. GHEORGHIU obtained inequalities (2.4.2) and (2.4.7) by the centroid method. There are no bibliographical references concerning the application of the centroid method to the inequalities in that paper.

In 1943, applying the centroid method, R. FRUCHT [**16**] proved inequality (2.4.3). He mentioned that he was inspired by the proof of inequalities

$$\sum_{i=1}^{n} a_i^2 \geqq \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2 \quad\text{and}\quad \sum_{i=1}^{n}\frac{1}{a_i} \geqq n^2\left(\sum_{i=1}^{n} a_i\right)^{-1}$$

given in the same Journal by E. M. SALEME [**15**]. R. FRUCHT does not cite any other literature. We now quote from the review of E. BECKENBACH: "The barycentric method previously has been used by Ş. A. GHEORGHIU (Bull. Math. Roum. Sci. **35**, 117—119 (1933)), to obtain the sharpened form of CAUCHY's inequality and also an analogously sharpened form of the HÖLDER-JENSEN inequality" (it seems that the word *sharpened* should be replaced by the word *complementary*). BECKENBACH did not mention earlier papers which employ the centroid method.

As we shall see in Section 3, inequalities (2.4.4) and (2.4.5) can also be proved by the centroid method.

2.5. Applying the centroid theorem M. TOMIĆ [**19**] proved the following theorem:

Let x_k and y_k $(k=1, \ldots, n)$ be real nondecreasing numbers from (a, b), i. e.

$$(2.5.1)\qquad x_k \leqq x_{k+1},\quad y_k \leqq y_{k+1} \qquad (k=1, \ldots, n)$$

and let f be a convex and decreasing function in (a, b). Inequality

$$(2.5.2)\qquad \sum_{i=1}^{n} f(x_i) \leqq \sum_{i=1}^{n} f(y_i)$$

holds provided that

$$(2.5.3)\qquad \sum_{i=1}^{k} y_i \leqq \sum_{i=1}^{k} x_i \qquad (k=1, \ldots, n).$$

We expose here the proof of M. TOMIĆ.

From (2.5.3) for $k=1$ follows $y_1 \leqq x_1$, and monotony of f implies

$$(2.5.4)\qquad f(y_1) \geqq f(x_1).$$

If $x_2 \leqq y_2$, denote by (A_1, B_1) and (a_1, b_1) the coordinates of the centroid T of $M_1 M_2$, and the centroid $T_1{}^*$ of $N_1 N_2$, where $M_1 = (y_1, f(y_1))$, $M_2 = (y_2, f(y_2))$, $N_1 = (x_1, f(x_1))$, $N_2 = (x_2, f(x_2))$. Then from (2.5.3) for $k=2$ we find

$$A_1 = \frac{y_1 + y_2}{2} \leqq \frac{x_1 + x_2}{2} = a_1,$$

and hence

$$B_1 = \frac{f(y_1) + f(y_2)}{2} \geqq \frac{f(x_1) + f(x_2)}{2} = b_1,$$

because in that case the segment $M_1 M_2$ is above the segment $N_1 N_2$, abscissa A_1 of T_1 is to the left from the abscissa of $T_1{}^*$, and hence the centroid T_1 is above the centroid $T_1{}^*$.

If $x_2 \geqq y_2$, then we have $f(y_2) \geqq f(x_2)$, which together with (2.5.4) yields

$$f(y_1) + f(y_2) \geqq f(x_1) + f(x_2). \tag{2.5.5}$$

Hence, in both cases inequality (2.5.5) is valid.

Apply the same procedure to the points T_1, M_3, $T_1{}^*$, N_3, where $M_3 = (y_3, f(y_3))$, $N_3 = (x_3, f(x_3))$. Then, if $x_3 \leqq y_3$, segment $T_1{}^* N_3$ is below the segment $T_1 M_3$. On the other hand, from (2.5.3) for $k=3$ we have

$$A_2 = \frac{y_1 + y_2 + y_3}{3} \leqq \frac{x_1 + x_2 + x_3}{3} = a_2.$$

Hence,

$$B_2 = \frac{f(y_1) + f(y_2) + f(y_3)}{3} \geqq \frac{f(x_1) + f(x_2) + f(x_3)}{3} = b_2.$$

If $x_3 \geqq y_3$, from the monotony of f follows $f(y_3) \geqq f(x_3)$, which together with (2.5.5) implies

$$f(y_1) + f(y_2) + f(y_3) \geqq f(x_1) + f(x_2) + f(x_3).$$

The theorem is now easily proved by mathematical induction.

2.6. H. Kestelman [**26**] proved the following result:

Suppose F is positive and continuous and G is positive and decreasing for $a \leqq x \leqq b$; then

$$\frac{\int_a^b x F(x) G(x) \, dx}{\int_a^b F(x) G(x) \, dx} < \frac{\int_a^b x F(x) \, dx}{\int_a^b F(x) \, dx}. \tag{2.6.1}$$

H. Kestelman mentioned the following: "Intuitively, the idea of the proof is that the centroid of the region under the curve $y = F(x) G(x)$ is closer to the y-axis than the centroid of the region under $y = F(x)$: this is to be expected because G decreases."

Remark 1. For $p(x) = f(x)$, $f(x) = x$, $g(x) = G(x)$ Čebyšev's inequality (see [**30**, p. 40], Theorem 10) yields inequality (2.6.1).

3. A COMPLEMENTARY INEQUALITY FOR JENSEN'S INEQUALITY

3.1. Let f be a given function such that $f(x) \geqq 0$ and $f''(x) \geqq 0$ on $[a, b]$. Let $P_1, \ldots, P_n$ be points with abscissae $x_1, \ldots, x_n$, ordinates $f(x_1), \ldots, f(x_n)$ and with masses $p_1, \ldots, p_n$. Let $m = \min x_i$, $M = \max x_i$. Then the set bounded by the arc of the curve $y = f(x)$ and the chord AB, where $A = (m, f(m))$, $B = (M, f(M))$ is convex.

From the family of curves $y = \lambda f(x)$ $(\lambda > 0)$ choose the curve which touches the segment AB. Then the centroid of the set of points $P_1, \ldots, P_n$ is below the arc of the curve $y = \lambda f(x)$ and above the arc of the curve $y = f(x)$, and hence

$$f\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i}\right) \leqq \frac{\sum_{i=1}^{n} p_i f(x_i)}{\sum_{i=1}^{n} p_i} \leqq \lambda f\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i}\right), \tag{3.1.1}$$

where λ is a constant to be determined.

The equation of the chord AB is

$$y = \frac{f(M) - f(m)}{M - m} x + \frac{M f(m) - m f(M)}{M - m}. \tag{3.1.2}$$

The conditions that the line (3.1.2) should be a tangent to $y = \lambda f(x)$ are given by

$$\lambda f(x) = \frac{f(M) - f(m)}{M - m} x + \frac{M f(m) - m f(M)}{M - m}, \tag{3.1.3}$$

$$\lambda f'(x) = \frac{f(M) - f(m)}{M - m}. \tag{3.1.4}$$

Elimination of λ from (3.1.3) and (3.1.4) yields

$$g(x) = \big(f(M) - f(m)\big) f(x) - f'(x) \Big(\big(f(M) - f(m)\big) x + M f(m) - m f(M)\Big) = 0.$$

Solution of the above equation is the abscissa of the contact point. We prove that this equation has exactly one solution $x_0 \in (m, M)$.

First, we have

$$g'(x) = -\Big(\big(f(M) - f(m)\big) x + M f(m) - m f(M)\Big) f''(x) = h(x) f''(x).$$

Now $f''(x) \geqq 0$, and the linear function h has

$$h(m) = (m - M) f(m) \leqq 0, \quad h(M) = (m - M) f(M) \leqq 0,$$

so provided $f''(x) > 0$, $m < M$ and $f(x) > 0$, the graph of g can cut the x-axis in at most one point of (m, M).

Furthermore,

$$g(m) g(M) = \left(f'(m) - \frac{f(M) - f(m)}{M - m}\right) \left(f'(M) - \frac{f(M) - f(m)}{M - m}\right) f(m) f(M) (M - m)^2,$$

and according to the mean value theorem and the fact that f' is an increasing function (since $f''(x)>0$), we have

$$g(m)\,g(M)\leq 0.$$

Therefore, there exist exactly one solution x_0 of the equation $g(x)=0$ on the interval (a, b), and so from (3.1.4) we obtain exactly one value for λ.

This value of λ is also the solution of the equation

$$\text{(3.1.5)}\quad \lambda f\left(f'^{-1}\left(\frac{f(M)-f(m)}{\lambda(M-m)}\right)\right)=\frac{f(M)-f(m)}{M-m}f'^{-1}\left(\frac{f(M)-f(m)}{\lambda(M-m)}\right)+\frac{Mf(m)-mf(M)}{M-m}.$$

Equality on the right side of (3.1.1) holds if and only if the point of contact of the segment AB and the curve $y=f(x)$ coincides with the centroid of the system of points $P_1, \ldots, P_n$. In order to obtain this situation, the centroid must lie on the straight line AB, which will happen if and only if $k\ (k<n)$ points among the points $P_1, \ldots, P_n$ coincide with A, and the remaining $n-k$ points coincide with B. Therefore, equality holds if and only if there exist two subsequences $(x_{i_1}, \ldots, x_{i_k})$ and $(x_{i_{k+1}}, \ldots, x_{i_n})$ of sequence $(x_1, \ldots, x_n)$ such that every element of the first subsequence is equal to m and every element of second subsequence is equal to M, and

$$x_0=\frac{m\sum_{r=1}^{k}p_{i_r}+M\sum_{r=k+1}^{n}p_{i_r}}{\sum_{i=1}^{n}p_i},\qquad f(x_0)=\frac{f(m)\sum_{r=1}^{k}p_{i_r}+f(M)\sum_{r=k+1}^{n}p_{i_r}}{\sum_{i=1}^{n}p_i},$$

where x_0 is a unique solution of the equation $g(x)=0$ on (a, b).

Remark 2. If $f(x)<0$ for $x\in S\subset[a, b]$, it is enough to consider the function F, defined by $F(x)=f(x)-\min\limits_{x\in[a,b]} f(x)$.

3.2. From the family of curves $y=\mu+f(x)$ $(\mu>0)$ choose the one which has a contact with the segment AB. Then the centroid of the system $P_1, \ldots, P_n$ is below the arc of the curve $y=\mu+f(x)$, and above the arc of the curve $y=f(x)$, and so

$$\text{(3.2.1)}\qquad f\left(\frac{\sum_{i=1}^{n}p_ix_i}{\sum_{i=1}^{n}p_i}\right)\leq\frac{\sum_{i=1}^{n}p_if(x_i)}{\sum_{i=1}^{n}p_i}\leq\mu+f\left(\frac{\sum_{i=1}^{n}p_ix_i}{\sum_{i=1}^{n}p_i}\right),$$

where μ is a constant to be determined.

The conditions that the line (3.1.2) should be a tangent to $y=\mu+f(x)$ are given by

$$\text{(3.2.2)}\qquad \begin{aligned}\mu+f(x)&=\frac{f(M)-f(m)}{M-m}x+\frac{Mf(m)-mf(M)}{M-m},\\ f'(x)&=\frac{f(M)-f(m)}{M-m}\end{aligned}$$

and we obtain the following value for μ:

$$\mu=\frac{f(M)-f(m)}{M-m}f'^{-1}\left(\frac{f(M)-f(m)}{M-m}\right)+\frac{Mf(m)-mf(M)}{M-m}-f\left(f'^{-1}\left(\frac{f(M)-f(m)}{M-m}\right)\right).$$

REMARK 3. In (3.1.1) we have assumed that $f(x)\geqq 0$. This supposition was necessary in order to prove the uniqueness of the solution of the equation $g(x)=0$. However, in this case, the uniqueness of the contact point follows from the second equality (3.2.2), since f' is an increasing function.

3.3. Consider the situation is which a mass is continuously distributed along an arc between the points A and B and suppose that the distribution of mass is defined by a function ρ. Then, the centroid coordinates are

$$x_T=\frac{\int_a^b \rho(x)\,x\sqrt{1+f'(x)^2}\,dx}{\int_a^b \rho(x)\sqrt{1+f'(x)^2}\,dx},\quad y_T=\frac{\int_a^b \rho(x)f(x)\sqrt{1+f'(x)^2}\,dx}{\int_a^b \rho(x)\sqrt{1+f'(x)^2}\,dx},$$

so that the following inequalities hold

$$f\left(\frac{\int_a^b \rho(x)\,x\sqrt{1+f'(x)^2}\,dx}{\int_a^b \rho(x)\sqrt{1+f'(x)^2}\,dx}\right)\leqq\frac{\int_a^b \rho(x)f(x)\sqrt{1+f'(x)^2}\,dx}{\int_a^b \rho(x)\sqrt{1+f'(x)^2}\,dx}\leqq\lambda f\left(\frac{\int_a^b \rho(x)\,x\sqrt{1+f'(x)^2}\,dx}{\int_a^b \rho(x)\sqrt{1+f'(x)^2}\,dx}\right),$$

where λ is the solution of (3.1.5), but with m, M replaced by a, b.

Given f (convex and nonnegative) and $g(x)>0$, choose $\rho(x)=g(x)\left(1+f'(x)^2\right)^{-1/2}$ to obtain

$$\text{(3.3.1)}\qquad f\left(\frac{\int_a^b x\,g(x)\,dx}{\int_a^b g(x)\,dx}\right)\leqq\frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}\leqq\lambda f\left(\frac{\int_a^b x\,g(x)\,dx}{\int_a^b g(x)\,dx}\right),$$

where λ is the solution of (3.1.5), with m, M replaced by a, b.

It should be noted that all results contained in paragraph **3** could be extended to convex functions of several variables.

REMARK 4. The left hand side of (3.3.1) has been derived by ST. I. GHEORGHITZA **[31]** by use of the centroid method.

4. SOME APPLICATIONS

4.1. Let $f(x)=x^2$ and consider the points with the abscissae $x_i=\frac{a_i}{b_i}$ $(i=1,\dots,n)$ and masses $p_ib_i^2$ $(i=1,\dots,n)$. Then, for $0<m_1\leqq a_i\leqq M_1$, $0<m_2\leqq b_i\leqq M_2$ $(i=1,\dots,n)$, it follows that

$$m=\min x_i=\min\frac{a_i}{b_i}\geqq\frac{\min a_i}{\max b_i}=\frac{m_1}{M_2},$$

$$M=\max x_i=\max\frac{a_i}{b_i}\leqq\frac{\max a_i}{\min b_i}=\frac{M_1}{m_2}.$$

Equation (3.1.5) gives

$$\lambda = \frac{(M+m)^2}{4\,mM} = \frac{\left(\frac{M}{m}+1\right)^2}{4\frac{M}{m}}.$$

In this case the inequality (3.1.1) reduces to

$$(4.1.1) \qquad \left(\sum_{i=1}^{n} p_i a_i^2\right)\left(\sum_{i=1}^{n} p_i b_i^2\right) \leqq \frac{(M+m)^2}{4Mm}\left(\sum_{i=1}^{n} p_i a_i b_i\right)^2,$$

where $m = \min \frac{a_i}{b_i}$, $M = \max \frac{a_i}{b_i}$.

However, $1 \leqq \frac{M}{m} \leqq \frac{M_1/m_2}{m_1/M_2} = \frac{M_1 M_2}{m_1 m_2}$. Setting

$$g(x) = \frac{(x+1)^2}{4x}, \quad g'(x) = \frac{1}{4}\left(1 - \frac{1}{x^2}\right),$$

we see that g is increasing for $x \geqq 1$. From this, it follows that

$$\lambda \leqq \frac{\left(\frac{M_1 M_2}{m_1 m_2}+1\right)^2}{4\frac{M_1 M_2}{m_1 m_2}} = \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2}.$$

Consequently, the inequality (4.1.1) is stronger than (2.4.4).

REMARK 5. For $p_i = 1$ $(i = 1, \ldots, n)$, the proof of the inequality (2.4.2) was given by Ş. A. GHEORGHIU **[12]** by use of the centroid method.

4.2. Let $f(x) = x^p$ $(p > 1)$ and consider the points having the abscissae $x_i = a_i b_i^{-q/p}$ $(i = 1, \ldots, n)$ and masses $b_i^q p_i$ $(i = 1, \ldots, n)$. Then, if $0 < m_1 \leqq a_i \leqq M_1$, $0 < m_2 \leqq \leqq b_i \leqq M_2$, we find

$$m = \min a_i b_i^{-q/p} \geqq (\min a_i)(\max b_i)^{-q/p} = m_1 M_2^{-q/p},$$

$$M = \max a_i b_i^{-q/p} \leqq (\max a_i)(\min b_i)^{-q/p} = M_1 m_2^{-q/p}.$$

In this case, the inequality (3.1.1) reduces to

$$(4.2.1) \qquad \left(\sum_{i=1}^{n} p_i a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} p_i b_i^q\right)^{\frac{1}{q}} \leqq \lambda^{\frac{1}{p}} \sum_{i=1}^{n} p_i a_i b_i,$$

where

$$(4.2.2) \qquad \lambda^{\frac{1}{p}} = q^{-\frac{1}{q}}\, p^{-\frac{1}{p}} \frac{M^p - m^p}{(M-m)^{1/p}(mM^p - Mm^p)^{1/q}}.$$

If function g, defined by

$$(4.2.3) \qquad g(x) = \frac{x^p - 1}{(x-1)^{1/p}(x^p - x)^{(p-1)/p}},$$

is increasing, for $x \geqq 1$, then inequality (4.1.1) is sharper than inequality (2.4.7). According to computations accomplished by D. V. SLAVIĆ on IBM 1130 it seems that this assumption is true. Naturally, it is to be analytically proved.

REMARK 6. For $p_i = 1$ $(i = 1, \ldots, n)$ the proof of the inequality (2.4.7) has been obtained by Ş. A. GHEORGHIU [12] by use of the centroid method.

4.3. Let $f(x) = x^{s/t}$ and consider the points with the abscissae a_i^t $(i = 1, \ldots, n)$ and masses p_i $(i = 1, \ldots, n)$. Then f is convex for $s > t > 0$ or $s > 0 > t$ and concave for $0 > s > t$. In the convex case, (3.1.1) becomes

$$\frac{\sum_{i=1}^{n} p_i a_i^s}{\sum_{i=1}^{n} p_i} \leqq \lambda \left(\frac{\sum_{i=1}^{n} p_i a_i^t}{\sum_{i=1}^{n} p_i} \right)^{s/t} \qquad \text{(with } \geqq \text{ in concave case).}$$

Since $s > 0$ in the convex case and $s < 0$ in concave case, this gives

$$\frac{M_n^{[s]}(a; p)}{M_n^{[t]}(a; p)} \leqq \lambda^{\frac{1}{s}}.$$

Now, (3.1.5) is the same as in **4.2.** but with $p = s/t$, so from (4.2.2) we obtain $\lambda^{1/s} = \Gamma_{s,t}$ in the case $s > t > 0$.

The proof of the inequality (2.4.5) in the cases $s > 0 > t$ and $0 > s > t$ is similar.

Now, let $t = 0$, and starting from the function f defined by $f(x) = -\log x$ and the points with the abscissae a_i^s $(i = 1, \ldots, n)$ and masses p_i $(i = 1, \ldots, n)$, we find, from (3.2.1)

$$\frac{M_n^{[s]}(a; p)}{M_n^{[0]}(a; p)} \leqq e^{\frac{\mu}{s}},$$

where μ is given by

$$\mu = -1 + \frac{m \log M - M \log m}{M - m} + \log \left(\frac{M - m}{\log M - \log m} \right),$$

where $m = m_1^s$, $M = M_1^s$. From this, we obtain $\Gamma_{s,0} = e^{\mu/s}$.

In the case $0 > t$ the proof is similar.

REMARK 7. Applying the centroid method to function f defined by $f(x) = -\log x$, K. DOČEV [28] obtained inequality (2.4.5) for $s = 1$, $t = 0$.

If we start from the function f defined by $f(x) = -\log x$ $(0 < x < 1)$ and the points with the abscissae a_i^s $(0 < a_i < 1)$ $(i = 1, \ldots, n)$ and the masses p_i $(i = 1, \ldots, n)$, then (3.1.1) reduces to

$$M_n^{[s]}(a; p) \leqq M_n^{[0]}(a; p)^{1/\lambda},$$

where λ is the unique solution of (3.1.5) which reduces to

$$\lambda \log \frac{\lambda(M-m)}{\log M - \log m} = \lambda + \frac{\log M^m - \log m^M}{M-m}$$

with $m = \min x_i = m_1^s$, $M = M_1^s$ $(s>0)$.

***

Professor P. R. BEESACK has read this paper and has given useful suggestions, particularly in connection with the applications presented in Section **4.**

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ADDED IN PROOF

1. While the paper was in print B. MESIHOVIĆ has proved that function g defined by (4.2.3), is increasing for $p>1$ and $x>1$. His proof reads:

Function g can be represented in the form

$$g(x)=u^{\frac{1}{p}}+u^{\frac{1}{p}-1}, \tag{1}$$

where

$$u(x)=\frac{x^p-x}{x-1}. \tag{2}$$

From (2) we get

$$\frac{du}{dx}=\frac{(p-1)x^p-px^{p-1}+1}{(x-1)^2}. \tag{3}$$

Consider the function h defined by

$$h(x)=(p-1)x^p-px^{p-1}+1.$$

Since for $p>1$ and $x>1$,

$$\frac{dh}{dx}=p(p-1)x^{p-2}(x-1)>0$$

and $h(1)=0$, we have $h(x)>0$ $(x>1)$, so that, on the basis of (3), function u is for $x>1$ increasing. Having in view that $\lim_{x \to 1+} u(x)=p-1$, we get

$$u(x)>p-1 \quad (x>1). \tag{4}$$

On the other hand

$$\frac{dg}{dx}=\frac{1}{p}u^{\frac{1}{p}-2}(u-p+1)\frac{du}{dx},$$

so that on the basis of (4) and the fact that u is an increasing function for $x>1$,

$$\frac{dg}{dx}>0,$$

which means that, for $x>1$, function g is increasing.

2. In connection with (2. 4. 4) see also a result of J. W. S. CASSELS given in G. S. WATSON: *Serial coorelations in regression analysis* I. Biometrika **42,** 327—341 (1955).