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496. INEQUALITIES INVOLVING ELEMENTS OF TRIANGLES, QUADRILATERALS OR TETRAHEDRA*

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Inequalities for pair of triangles and quadrilaterals

Theorem 1. *Suppose that* $A_i B_i C_i$ $(i=1, 2)$ *are two triangles of sides* a_i, b_i, c_i, *areas* F_i, *circumradii* R_i. *Construct a third triangle by taking*

$$a_3=(a_1{}^2+a_2{}^2)^{1/2}, \quad b_3=(b_1{}^2+b_2{}^2)^{1/2}, \quad c_3=(c_1{}^2+c_2{}^2)^{1/2}.$$

Then we have

$$R_3{}^2 \leqq R_1{}^2+R_2{}^2. \tag{1}$$

Equality in (1) *occurs if either the two given triangles are similar or if the two triangles are right-angled at corresponding vertices.*

Proof. I give a proof of (1) by considering the stationary values of

$$E=R_1{}^2+R_2{}^2-R_3{}^2$$

subject to variation in a_1, b_1, c_1, the second triangle being fixed. Elementary calculation gives

$$\frac{\partial R_1{}^2}{\partial a_1}=\frac{2\,a_1\,b_1{}^2\,c_1{}^2}{T_1}-\frac{a_1{}^2\,b_1{}^2\,c_1{}^2}{T_1^2}\,4\,a_1\,(b_1{}^2+c_1{}^2-a_1{}^2)$$

where $T_1=16\,F_1{}^2$. Thus

$$\frac{T_1^2}{2\,a_1}\,\frac{\partial R_1{}^2}{\partial a_1}=b_1{}^2\,c_1{}^2\left\{\sum a_1{}^2\,(b_1{}^2+c_1{}^2-a_1{}^2)-2\,a_1{}^2\,(b_1{}^2+c_1{}^2-a_1{}^2)\right\}$$
$$=b_1{}^2\,c_1{}^2\,(c_1{}^2+a_1{}^2-b_1{}^2)\,(a_1{}^2+b_1{}^2-c_1{}^2).$$

Since $a_3\,\partial a_3/\partial a_1=a_1$, the condition $\partial E/\partial a_1=0$ becomes

$$T_1{}^{-2}\,b_1{}^2\,c_1{}^2\,(c_1{}^2+a_1{}^2-b_1{}^2)\,(a_1{}^2+b_1{}^2-c_1{}^2)$$
$$=b_3{}^2\,c_3{}^2\,(c_3{}^2+a_3{}^2-b_3{}^2)\,(a_3{}^2+b_3{}^2-c_3{}^2)\,T_3{}^{-2};$$

two like equations arise by cyclic permutation of the letters.

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If the angle C_1 is a right angle then either C_3 is a right angle (and then also C_2) or else B_3 is a right angle. But in the last case the second equation shows that A_3 is also a right angle which cannot occur for non-degenerate triangles.

If $C_1 = C_3 = \pi/2$, all three equations are satisfied and indeed $R_1{}^2 + R_2{}^2 - R_3{}^2 = 0$, no other condition on the sides being required.

Right-angled triangles being excluded, the three equations show that

$$\frac{\sin 2A_1}{\sin 2A_3} = \frac{\sin 2B_1}{\sin 2B_3} = \frac{\sin 2C_1}{\sin 2C_3} = \frac{1}{k}$$

where $k \neq 0$. Denote the sines of these angles by u, v, w, u', v', w' respectively. Then

$$\sum u^4 - 2\sum v^2 w^2 + 4u^2 v^2 w^2 = 0,$$

$$\sum u'^4 - 2\sum v'^2 w'^2 + 4u'^2 v'^2 w'^2 = 0,$$

so that also

$$k^4 \left[\sum u^4 - 2\sum v^2 w^2 + 4k^2 u^2 v^2 w^2\right] = 0.$$

Hence $(k^2 - 1)\, u^2 v^2 w^2 = 0$ so that since $uvw \neq 0$ (right angled triangles have been excluded) we have $k = \pm 1$.

It is now easy to see that only one possibility remains,

$$A_1 = A_3, \quad B_1 = B_3, \quad C_1 = C_3,$$

i.e. triangles are similar.

(If $k = 1$, we have

$$2A_1 = 2A_3 \text{ or } \pi - 2A_3, \quad 2B_1 = 2B_3 \text{ or } \pi - 2B_3, 2C_1 = 2C_3 \text{ or } \pi - 2C_3.$$

A combination such as

$$2A_1 = \pi - 2A_3, \quad 2B_1 = 2B_3, \quad 2C_1 = 2C_3$$

yields by addition

$$2\pi = \pi + 2\pi - 2A_3; \; A_3 = \pi/4; \; A_1 = \pi/4, \; B_1 = B_3, \; C_1 = C_3.$$

The combination $2A_1 = \pi - 2A_3$, $2B_1 = \pi - 2B_3$, $2C_1 = 2C_3$ yields

$$2\pi = 2\pi - 2\pi + 4C_3; \; C_3 = \pi/2 = C_1.$$

The case $k = -1$ is equally easy to settle).

Since E does take positive values and for these points $E = 0$ (and no other stationary points arise) we must have $E \geqq 0$.

REMARK 1. The inequality $R_1{}^2 + R_2{}^2 \geqq R_3{}^2$ gives rise to a curious unsymmetric inequality between two triangles:

(2) $$16\, a_2{}^2 b_2{}^2 c_2{}^2 F_1{}^4 + 2\, a_1{}^2 b_1{}^2 c_1{}^2 F_2{}^2 \sum a_1{}^2 (b_2{}^2 + c_2{}^2 - a_2{}^2) - 16\, F_1{}^2 F_2{}^2 \sum b_1{}^2 c_1{}^2 a_2{}^2 \geqq 0.$$

Equality occurs if the two triangles are similar or if

$$A_1 = A_2 = \pi/2 \text{ or if } B_1 = B_2 = \pi/2 \text{ or if } C_1 = C_2 = \pi/2$$

and in no other case.

It will be recalled that

$$\sum a_1^2 (b_2^2 + c_2^2 - a_2^2) \geqq 16 F_1 F_2 \qquad \text{(PEDOE, [1] 10.8)} \tag{3}$$

equality only for similar triangles.

If we replace the left term of (3) by $16 F_1 F_2$ in (2) we obtain an inequality going *the other way*:

$$16 a_2^2 b_2^2 c_2^2 F_1^4 + 32 a_1^2 b_1^2 c_1^2 F_1 F_2^3 - 16 F_1^2 F_2^2 \sum b_1^2 c_1^2 a_2^2 \leqq 0, \tag{4}$$

zero if and only if the two triangles are similar.

Suppose that ABC is a triangle, with sides a, b, c and area F. Then $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ are also the sides of a triangle with area H such that

$$4 H^2 \geqq F \sqrt{3} \tag{5}$$

with equality if and only if ABC is equilateral.

(FINSLER — HADWIGER, 1937/38: see the book *Geometric Inequalities*, **[1]**, **10.3** and references therein.)

It is reasonable to ask whether a corresponding inequality holds with exponent $1/2$ replaced by $1/p$ where $p > 1$. We prove in fact:

Theorem 2. *If ABC is a triangle of sides a, b, c and area F, there exists a triangle of sides $a^{1/p}$, $b^{1/p}$, $c^{1/p}$, $(p > 1)$ and area F_p such that*

$$(4 F_p/\sqrt{3})^p \geqq 4 F/\sqrt{3}. \tag{6}$$

Equality holds only if $a = b = c$.

In other words $(4 F_p/\sqrt{3})^p$ is an increasing function of p (bounded of course by $(abc)^{2/3}$).

The corresponding circumradii satisfy the inequality

$$(R_p \sqrt{3})^p \leqq R \sqrt{3}. \tag{7}$$

$(R_p \sqrt{3})^p$ is a decreasing function of p, bounded below by 1.

It is known also that if $f(x)$ is a non-negative, non-decreasing sub-additive function on $x > 0$ then a triangle with sides a, b, c will yield a triangle with sides $f(a)$, $f(b)$, $f(c)$, ([**1**], **13.3**). It is natural to conjecture an inequality for the corresponding areas.

Conjecture. *Let $G(a, b, c) = (4 F(a, b, c)/\sqrt{3})^{1/2}$. Then*

$$G(f(a), f(b), f(c)) \geqq f(G(a, b, c)). \tag{8}$$

Equality holds in general only when $a = b = c$.

(I say in general since f may be a constant function.)

I have not been able to prove (8) but *if* (8) *holds for two particular such functions then* (8) *will hold for their sum.*

The last statement derives from the case $p = 1$ of the following

Theorem 3. ([2] Theorem 6). *Suppose that $A_i B_i C_i$ are two triangles* ($i=1, 2$). *Define for any $p>1$, $a=(a_1{}^p+a_2{}^p)^{1/p}$, ... Then a, b, c are the sides of a triangle. The three areas are connected by the inequality (if $p=1$ or 2 or 4)*

$$F^{p/2} \geqq F_1{}^{p/2}+F_2{}^{p/2};$$

equality if and only if the triangles are similar.

As shown in [2] the inequality does not hold for $p>4$. It is not known whether it holds for $1 \leqq p \leqq 4$ (other than 1, 2, 4).

***Proof of Theorem* 2.** To ease the writing suppose that ABC has sides a^p, b^p, c^p so that the second triangle has sides a, b, c. It is then a queston of proving that

$$E=U^p-3^{p-1}\left(2\sum b^{2p}c^{2p}-\sum a^{4p}\right) \tag{9}$$

where $U=2\sum b^2c^2-\sum a^4$ has minimum 0, attained for $a=b=c$.

Since E is homogeneous it is enough to determine stationary values subject to $abc=$const. Partial differentiation and EULER's theorem on homogeneous functions yield the conditions

$$a\frac{\partial E}{\partial a}=b\frac{\partial E}{\partial b}=c\frac{\partial E}{\partial c}=\frac{4}{3}pE$$

whence

$$3a^2(b^2+c^2-a^2)U^{p-1}-3^p a^{2p}(b^{2p}+c^{2p}-a^{2p})=E \tag{10}$$

and two like equations (11), (12) by cyclic permutation of a, b, c.

One solution of these three equations is plainly $a=b=c$ for which $E=0$. If a different solution exists we may suppose by symmetry that

$$a>c\geqq b \text{ or } a\geqq c>b.$$

From (10) and (11) by subtraction,

$$(a^2-b^2)(a^2+b^2-c^2)U^{p-1}=3^{p-1}(a^{2p}-b^{2p})(a^{2p}+b^{2p}-c^{2p}). \tag{13}$$

Eliminate U^{p-1} between (12) and (13), we find that

$$(a^2-b^2)E=3^p c^p(a^{2p}+b^{2p}-c^{2p})\{a^2(a^{2p-2}-c^{2p-2})+b^2(c^{2p-2}-b^{2p-2})\}$$

which shows that $E>0$.

Thus theorem 2 follows: equality holds only for equilateral triangles.

Inequality for pair of quadrilaterals

Theorem 2 does not extend to convex cyclic quadrilaterals but Theorem 3 does.

Theorem 4. *Suppose that $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ are two convex cyclic quadrilaterals of sides $a_1, \ldots, d_1$ and $a_2, \ldots, d_2$. Define $a, \ldots,$ by*

$$a=(a_1{}^p+a_2{}^p)^{1/p} \qquad (p>1).$$

Then there is a convex cyclic quadrilateral of sides $a, \ldots$; the areas satisfy (for $p=1, 2, 4$) the inequality

$$F^{p/2}\geqq F_1{}^{p/2}+F_2{}^{p/2},$$

equality if and only if the given polygons are similar.

As in the case of Theorem 3 [2] the proofs for $p=1, 2, 4$ are different.

Proof for $\boldsymbol{p=1}$**.** The area of a convex cyclic quadrilateral of edges a, b, c, d is given by $16F^2=uvwt$ where $u=-a+b+c+d$, $v=a-b+c+d, \dots$

Thus

$$2F^{1/2}=(uvwt)^{1/4}$$

where

$$u=u_1+u_2, \quad v=v_1+v_2, \dots$$

A well-known inequality gives

$$(uvwt)^{1/4} \geqq (u_1 v_1 w_1 t_1)^{1/4}+(u_2 v_2 w_2 t_2)^{1/4}$$

(strict unless $u_1:v_1:w_1:t_1=u_2:v_2:w_2:t_2$). Hence

$$F^{1/2} \geqq F_1^{1/2}+F_2^{1/2}$$

as required.

Proof for $\boldsymbol{p=4}$**.** Here we use the formula

$$16F^2=2\sum a^2b^2-\sum a^4+8abcd.$$

Hence

$$\begin{aligned}16F^2-16F_1^2-16F_2^2 &= 2\sum (a_1^4+a_2^4)^{1/2}(b_1^4+b_2^4)^{1/2}-\sum (a_1^4+a_2^4)\\ &\quad -2\sum a_1^2b_1^2-2\sum a_2^2b_2^2+\sum a_1^4+\sum a_2^4\\ &\quad +8\prod (a_1^4+a_2^4)^{1/4}-8\left(\prod a_1+\prod a_2\right)\\ &= 2\sum \{(a_1^4+a_2^4)^{1/2}(b_1^4+b_2^4)^{1/2}-(a_1^2b_1^2+a_2^2b_2^2)\}\\ &\quad +8\left(\prod a_1^4+a_2^4\right)^{1/4}-8\left(\prod a_1+\prod a_2\right)\end{aligned}$$

which is positive or zero by standard inequalities.

Proof for $\boldsymbol{p=2}$. We use the result of Theorem 3 for the triangle in the following way. From $A_1B_1C_1D_1$ we get two triangles $A_1B_1C_1$ with sides $a_1=A_1B_1$, $b_1=B_1C_1$ and $x_1=C_1A_1$; $A_1C_1D_1$ with sides $x_1=A_1C_1$, $c_1=C_1D_1$, $d_1=D_1A_1$. Like-wise from $A_2B_2C_2D_2$ two triangles.

Using triangles $A_1B_1C_1$, $A_2B_2C_2$ we derive a triangle ABC; areas satisfy

$$F(a, b, x) \geqq F(a_1, b_1, x_1)+F(a_2, b_2, x_2);$$

so from triangles $A_1C_1D_1$, $A_2C_2D_2$ we derive a triangle ACD: areas satisfy

$$F(x, c, d) \geqq F(x_1, c_1, d_1)+F(x_2, c_2, d_2).$$

The triangles $A_1C_1D_1$, $A_1B_1C_1$ fit together to give convex cyclic quadrilateral $A_1B_1C_1D_1$; area F_1. So $A_2C_2D_2$, $A_2B_2C_2$ give $A_2B_2C_2D_2$ of area $\boldsymbol{F_2}$. Thus $F(a, b, x)+F(x, c, d) \geqq F_1+F_2$.

Now the sides a, b, c, d (from the two triangles ABC, ACD) yield a convex cyclic quadrilateral of area $\geqq F(a, b, x)+F(x, c, d)$. Thus $F \geqq F_1+F_2$; equality holds if the original quadrilaterals are similar.

The proof of Theorem 4 is complete.

REMARK 2. That Theorem 4 can be extended when $p=2$ to convex cyclic n-gons seems clear. Whether Theorem 4 has such an extension for $p=1$, $p=4$ is doubtful.

Inequalities for Tetrahedra

The equalities and inequalities obtained below for tetrahedra are interesting: so far as I can ascertain they are new.

Suppose that $ABCD$ is a tetrahedron of volume V, circumradius R; a, b, c edges of the face ABC; p, q, r the opposite edges AD, BD, CD. Then

$$6\,RV = F(ap, bq, cr) \tag{14}$$

where $F(u, v, w)$ denotes the area of the triangle of sides u, v, w;

$$64\,R^4 \geqq (a^2+b^2+c^2)\,(p^2+q^2+r^2) \tag{15}$$

with equality if and only if $p=a$, $q=b$, $r=c$, i.e. the tetrahedron has congruent faces (necessarily acute-angled).

From (14) follows the inequality

$$6\,R_0\,V \leqq F(ap, bq, cr)$$

where R_0 is the circumradius of the triangle ABC; equality if and only if ABC is a great circle of the circumscribing sphere. If in this inequality we take ABC to be equilateral, then

$$12\,V^2 \leqq a^2\,F(p, q, r),$$

an inequality due to BOTTEMA (see [1], **12.4**).

From (14) and (15) we obtain

$$72\,V^2 \leqq \frac{\Sigma\,ap}{(\Sigma\,a^2)^{1/2}\,(\Sigma\,p^2)^{1/2}} \prod\,(bq+cr-ap) \tag{16}$$

and so by the CAUCHY-SCHWARZ inequality

$$72\,V^2 \leqq (bq+cr-ap)\,(cr+ap-bq)\,(ap+bq-cr); \tag{17}$$

equality in (16) or (17) only for tetrahedra with congruent faces.

The equality (14) may be regarded as the analogue of the familiar triangle identity $4\,R_0\,F(a, b, c)=abc$ while the inequality (15) is an analogue of the inequality $9\,R_0{}^2 \geqq a^2+b^2+c^2$, equality only for the equilateral triangle.

Proof that $6\,RV=F(ap, bq, cr)$.

The simplest proof comes by inverting with respect to D the relation

$$3\,V = h\,F(a, b, c)$$

where h is the altitude from D to the face ABC.

For, if accented letters refer to the inverses of ABC and if $B'C'=a'$, $DA'=p'$ and so on, then

$$pp'=k^2, \quad a'/a=k^2/qr, \quad (2\,R)\,h'=k^2, \quad V'/V=p'q'r'/pqr.$$

Hence $3\,V'=h'\,F(a', b', c')$ becomes

$$3\frac{p'q'r'}{pqr}V=\frac{k^2}{2\,R}F\left(\frac{k^2\,a}{qr},\ \ldots\right)=\frac{k^6}{2\,Rp^2q^2r^2}F(ap, bq, cr)$$

or

$$6\,RV=F(ap, bq, cr)$$

which is (14). (The proof shows incidentally that ap, bq, cr are the sides of a triangle.)

Proof that $64\,R^4 \geqq (a^2+b^2+c^2)(p^2+q^2+r^2)$ with equality if and only if $a=p,\ b=q,\ c=r$.

Take O, centre of sphere, as the origin of vectors; take $R=1$ so that if $OA,\ OB,\ OC,\ OD$ are the unit vectors $\alpha,\ \beta,\ \gamma,\ \delta$ we have

$$p^2 = \sum (\delta-\alpha)^2 = 6 - 2\,\delta\cdot(\alpha+\beta+\gamma).$$

Thus, for given $\alpha,\ \beta,\ \gamma,\ \sum p^2$ is maximised when

$$\delta = -t\,(\alpha+\beta+\gamma), \quad t>0.$$

(DOG are collinear, G centroid of ABC; O between D and G). Note that

$$1 = 1\,\delta\,1 \leqq t \sum 1\,\alpha\,1 = 3\,t.$$

Also

$$\sum a^2 = \sum (\beta-\gamma)^2 = 6 - 2\sum \beta\cdot\gamma,$$

$$-\delta\cdot\sum\alpha = t\left(\sum\alpha\right)^2 = t\left(3+2\sum\beta\cdot\gamma\right),$$

$$1 = t^2\left(\sum\alpha\right)^2 = t^2\left(3+2\sum\beta\cdot\gamma\right).$$

Hence

$$\sum a^2 \sum p^2 = (9-1/t^2)\,(6+2/t) \leqq 64$$

for $1 \leqq 3\,t$; attained when $t=1$. But when $t=1$

$$\delta+\alpha = -\beta-\gamma, \quad 2+\delta\cdot\alpha = 2+\beta\cdot\gamma,$$

$$p^2 = (\delta-\alpha)^2 = (\beta-\gamma)^2 = a^2.$$

Thus $\sum a^2 \sum p^2 \leqq 64$, equality if and only if opposite edges are equal.

REFERENCES

1. O. Bottema, R. Ž. Đorđević, R. R. Janić, D. S. Mitrinović, P. M. Vasić: *Geometric Inequalities*. Groningen, 1969.

2. A. Oppenheim: *Some inequalities for triangles*. These Publications **№ 357 — № 380** (1971), 21—28.

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