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494. FUNCTIONAL EQUATIONS FOR WALLIS AND GAMMA FUNCTIONS*

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The aim of this note is to find all convex solutions of the functional equation

$$f(x+1)=\frac{x+1}{x+\theta}f(x), \qquad x\in[0,\,+\infty) \tag{1}$$

where $f:\mathbf{R}_+\to\mathbf{R}$, $\mathbf{R}_+=[0,\,+\infty)$ and θ is a prescribed number on $(0,\,1)$. Setting $f(0)=1/\Gamma(\theta)$ we shall see that the unique solution is the „Wallis function" $W(\cdot,\,\theta):\mathbf{R}_+\to\mathbf{R}$ defined as

$$W(x,\,\theta)=\frac{\Gamma(x+1)}{\Gamma(x+\theta)}.$$

Further we establish some inequalities for the Wallis function. At the end of this paper a new characterization of the Gamma function through functional equation is given.

Theorem 1. *Let* $A\in\mathbf{R}_+$, $\alpha\in(0,\,+\infty)$, $\theta\in(0,1)$ *be fixed elements. There is a unique solution* $f:\mathbf{R}_+\to\mathbf{R}$ *of* (1) *defined by*

$$f(x)=\alpha\frac{\Gamma(\theta)\,\Gamma(x+1)}{\Gamma(x+\theta)}$$

which is convex on $(A,\,+\infty)$ *and such that* $f(0)=\alpha$.

Proof. It is easy to see that for a natural number n

$$f(x+n)=\frac{(x+n)(x+n-1)\cdots(x+1)}{(x+n+\theta-1)(x+n+\theta-2)\cdots(x+\theta)}f(x), \qquad x\in\mathbf{R}_+$$

or

$$f(x+n)=\frac{\Gamma(x+n+1)\,\Gamma(x+\theta)}{\Gamma(x+n+\theta)\,\Gamma(x+1)}f(x), \qquad x\in\mathbf{R}_+. \tag{2}$$

Therefore

$$f(y)=f([y]+\{y\})=\frac{\Gamma(y+1)\,\Gamma(\{y\}+\theta)}{\Gamma(y+\theta)\,\Gamma(\{y\}+1)}f(\{y\}), \qquad y\in[1,\,+\infty)$$

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which confirms that it is sufficient to suppose $x \in [0, 1)$. Let $n \geqq 2 + [A]$. Since f is convex (non-concave) on $(A, +\infty)$ we may write

$$[n-1, n; f] \leqq [n, n+x; f] \leqq [n, n+1; f], \quad x \in (0, 1),$$

where the symbol $[a, b; f]$ denotes the divided difference.

Using (1) and the above inequalities one obtains

$$\frac{1-\theta}{n} f(n) \leqq \frac{f(n+x)-f(n)}{x} \leqq \frac{1-\theta}{n+\theta} f(n)$$

which can be written as

$$\frac{n+x(1-\theta)}{n} f(n) \leqq f(n+x) \leqq \frac{n+\theta+x(1-\theta)}{n+\theta} f(n).$$

But from these inequalities as well as from (2), for $x \in (0, 1)$ we have

$$\frac{n+x(1-\theta)}{n} f(n) \leqq \frac{\Gamma(n+x+1)\,\Gamma(x+\theta)}{\Gamma(n+x+\theta)\,\Gamma(x+1)} f(x) \leqq \frac{n+\theta+x(1-\theta)}{n+\theta} f(n).$$

From these inequalities we may write

$$\frac{\alpha(n+x(1-\theta))\,\Gamma(\theta)\,\Gamma(x+1)}{n\,\Gamma(x+\theta)} F_n(x, \theta) \leqq f(x) \tag{3}$$

$$\leqq \frac{\alpha(n+\theta+x(1-\theta))\,\Gamma(\theta)\,\Gamma(x+1)}{(n+\theta)\,\Gamma(x+\theta)} F_n(x, \theta)$$

where

$$F_n(x, \theta) = \frac{\Gamma(n+1)}{\Gamma(n+\theta)} \cdot \frac{\Gamma(n+x+\theta)}{\Gamma(n+x+1)}.$$

We remark that (3) is trivially verified at the point $x=0$, i.e., the inequalities (3) are valid on $[0, 1)$.

According to a well-known theorem by H. Bohr and I. Mollerup **[3]** (see also **[1–2]**) the restriction at $(0, +\infty)$ of the Gamma function, is the unique logarithmic-convex function on $(B, +\infty)$, $B \geqq 0$, which satisfies

$$\Gamma(x+1) = x\,\Gamma(x), \quad \Gamma(1) = 1.$$

This means that for $0 < a < b < c < +\infty$ the inequality

$$[a, b, c; \ln \Gamma(\cdot)] > 0$$

implies

$$(\Gamma(b))^{c-a} < (\Gamma(a))^{c-b} (\Gamma(c))^{b-a}.$$

Setting $a = z$, $b = z + \xi$, $c = z + 1$, $\xi \in (0, 1)$, we get

$$\frac{(\Gamma(z+1))^{\xi}}{\Gamma(z+\xi)} > (\Gamma(z))^{\xi-1},$$

or

$$\frac{\Gamma(z+1)}{\Gamma(z+\xi)} > z^{1-\xi}, \quad z \in (0, 1), \tag{4}$$

which holds also at $z=0$.

On the other hand the following may be proved (see [**4**], Lemma 1): If $g:\mathbf{R}_+\times(0,1)\to(0,+\infty)$ is such that for all $(z,\theta)\in\mathbf{R}_+\times(0,1)$

$$\frac{\Gamma(z+1)}{\Gamma(z+\theta)}>g(z,\theta),$$

then

$$\frac{\Gamma(z+1)}{\Gamma(z+\theta)}<\frac{z+\theta}{g(z+\theta,\,1-\theta)},\qquad \theta\in(0,1).$$

In this manner, by means of (4), we conclude with the inequalities

$$z^{1-\theta}<\frac{\Gamma(z+1)}{\Gamma(z+\theta)}<(z+\theta)^{1-\theta},\qquad (z,\theta)\in\mathbf{R}_+\times(0,1). \tag{5}$$

Put $z=n$, $z=n+x$ respectively in (5); we find

$$n^{1-\theta}<\frac{\Gamma(n+1)}{\Gamma(n+\theta)}<(n+\theta)^{1-\theta}$$

and

$$\frac{1}{(n+x+\theta)^{1-\theta}}<\frac{\Gamma(n+x+\theta)}{\Gamma(n+x+1)}<\frac{1}{(n+x)^{1-\theta}}.$$

From these inequalities as well as from (2), by mutual multiplications of the corresponding members we get

$$\left(\frac{n}{n+x+\theta}\right)^{1-\theta}<F_n(x,\theta)<\left(\frac{n+\theta}{n+x}\right)^{1-\theta},\qquad (x,\theta)\in\mathbf{R}_+\times(0,1).$$

In conclusion, (3) implies that for $x\in\mathbf{R}_+$, $\theta\in(0,1)$,

$$\frac{\alpha(n+x(1-\theta))}{n}\left(\frac{n}{n+x+\theta}\right)^{1-\theta}\cdot\frac{\Gamma(\theta)\Gamma(x+1)}{\Gamma(x+\theta)}\leqq f(x)$$

$$\leqq\frac{\alpha(n+\theta+x(1-\theta))}{n+\theta}\left(\frac{n+\theta}{n+x}\right)^{1-\theta}\cdot\frac{\Gamma(\theta)\Gamma(x+1)}{\Gamma(x+\theta)}.$$

For $n\to+\infty$ the general convex solution of (1)

$$f(x)=\alpha\frac{\Gamma(\theta)\Gamma(x+1)}{\Gamma(x+\theta)}$$

is found and the theorem is proved thereby.

We note that a similar functional equation was treated by J. ANASTASSIADIS [**1**] by a different method and supposing that f is positive and logarithmic-convex.

Theorem 2. *Let $W(\cdot,\theta):\mathbf{R}_+\to\mathbf{R}_+$ be the Wallis function. There exists a decreasing, convex function $\varepsilon_\theta:\mathbf{R}_+\to(a,b]$ where*

$$a=\frac{\theta}{2},\quad b=(\Gamma(\theta))^{\frac{1}{\theta-1}},\qquad \theta\in(0,1)$$

such that

$$W(x,\theta)=(x+\varepsilon_\theta(x))^{1-\theta},\qquad x\in\mathbf{R}_+. \tag{6}$$

Proof. On account of (5) we observe that there is a function $\varepsilon_\theta : \mathbf{R}_+ \to \mathbf{R}$ with

$$W(x, \theta) = (x + \varepsilon_\theta(x))^{1-\theta}, \quad 0 < \varepsilon_\theta(x) < \theta, \quad x \in \mathbf{R}_+.$$

Since

$$\varepsilon_\theta(x) = W(x, \theta)^{\frac{1}{1-\theta}} - x, \tag{7}$$

it is clear that $\varepsilon_\theta : \mathbf{R}_+ \to (0, \theta)$ is convex on its domain. Indeed according to the first theorem $W(\cdot, \theta)$ is convex on $\mathbf{R}_+$. On the other hand, for $x > 0$ we have

$$\varepsilon_\theta'(x) = \frac{x + \varepsilon_\theta(x)}{1-\theta} \left(\frac{\Gamma'(x+1)}{\Gamma(x+1)} - \frac{\Gamma'(x+\theta)}{\Gamma(x+\theta)} \right) - 1$$

$$< (x+\theta)\,\psi'(x+\theta) - 1 < \frac{1}{x+\theta-1}.$$

where $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$. Further, ε_θ' is increasing on $\mathbf{R}_+$ and

$$\varepsilon_\theta'(x) \leqq \lim_{x \to +\infty} \varepsilon_\theta'(x) \leqq 0,$$

that is ε_θ is a decreasing function. This implies

$$\varepsilon_\theta(x) \leqq \varepsilon_\theta(0), \qquad x \in \mathbf{R}_+,$$

with equality only for $x = 0$. From (7), $\varepsilon_\theta(0) = (\Gamma(\theta))^{\frac{1}{1-\theta}}$, and in this way

$$\varepsilon_\theta(x) \leqq b, \qquad x \in \mathbf{R}_+.$$

Finally, by means of STIRLING series we prove that

$$\lim_{x \to +\infty} \varepsilon_\theta(x) = \frac{\theta}{2},$$

i.e.,

$$a = \frac{\theta}{2} = \lim_{x \to +\infty} \varepsilon_\theta(x) < \varepsilon_\theta(x), \qquad x \in \mathbf{R}_+,$$

which completes the proof.

Corollary. *For* $(x, \theta) \in \mathbf{R}_+ \times (0, 1)$ *we have*

$$\left(x + \frac{\theta}{2}\right)^{1-\theta} < \frac{\Gamma(x+1)}{\Gamma(x+\theta)} < \left\{ x + (\Gamma(\theta))^{\frac{1}{1-\theta}} \right\}^{1-\theta}.$$

The case $\theta = 1/2$ leads to G. N. WATSON's result [**6**], namely

$$\left(x + \frac{1}{4}\right)^{1/2} < \frac{\Gamma(x+1)}{\Gamma\left(x + \frac{1}{2}\right)} \leqq \left(x + \frac{1}{\pi}\right)^{1/2}, \qquad x \in \mathbf{R}_+.$$

For other information regarding the inequalities involving Gamma function see [**5**].

Now we intend to find all solutions of the functional equation

$$f(x+1) = x f(x), \qquad x \in (0, +\infty)$$

in some classes of functions.

Theorem 3. *There exists a unique function* $f:(0, +\infty) \to \mathbf{R}$, *positive on* $(0, 1)$ *and satisfying*:

(i) $f(x+1) = xf(x), \qquad x \in (0, +\infty)$;

(ii) f *is logarithmic-concave of the second order on* $(A, +\infty)$, A *being a prescribed non-negative number*;

(iii) $f(1) = 1$.

More precisely, this function coincides with the restriction at $(0, +\infty)$ *of the Gamma function.*

Proof. If n is a natural number, then any solution of (i) has the property

$$f(n+x) = x(x+1)\cdots(x+n-1)f(x). \tag{8}$$

Let $y \in (1, +\infty)$, $y = [y] + \{y\}$; it is clear that

$$f(y) = \{y\}(\{y\}+1)\cdots(y-1)f(\{y\}).$$

Therefore the positivity of f at $(0, 1)$ implies that $f:(0, +\infty) \to \mathbf{R}$ is positive on its domain.

As usual, a logarithmic-concave function f of the second order on $(A, +\infty)$, has the properties: f is positive on $(A, +\infty)$ and $[a, b, c, d; \ln f] < 0$ for any points $A < a < b < c < d < +\infty$, which may be written as

$$\big(f(b)\big)^{(d-a)(c-a)(d-c)} \cdot \big(f(d)\big)^{(c-b)(b-a)(c-a)} < \big(f(a)\big)^{(c-b)(d-c)(d-b)} \cdot \big(f(c)\big)^{(d-a)(d-b)(b-a)}. \tag{9}$$

If we select

$$a = n-1, \; b = n+x-1, \; c = n, \; d = n+x, \; x \in (0, 1), \; n \geqq 2 + [A]$$

we obtain

$$(f(n+x-1))^{1+x}(f(n+x))^{1-x} < (f(n-1))^{1-x}(f(n))^{1+x}. \tag{10}$$

Taking into account (i) as well as the fact that $f(n) = (n-1)!$ and $f(n+x-1) = \frac{1}{n+x-1} f(n+x)$, from (10) we get

$$[f(n+x)]^2 < ((n-1)!)^2 \frac{(n-x-1)^{1+x}}{(n-1)^{1-x}}$$

$$= ((n-1)!)^2 n^{2x} \left(1 + \frac{1}{n-1}\right)^{1-x} \left(1 + \frac{1-x}{n+x-1}\right)^{-1-x}.$$

On the other hand [5, p 262, § 3.6.3] we have

$$\left(1 + \frac{1-x}{n+x-1}\right)^{1+x} > e^{\frac{2(1-x^2)}{2n+x-1}}, \quad \left(1 + \frac{1}{n-1}\right)^{1-x} < e^{\frac{1-x}{n-1}}.$$

Therefore

$$f(n+x)<n^x(n-1)!\sqrt{\left(1+\frac{1}{n-1}\right)^{1-x}\left(1+\frac{1-x}{n+x-1}\right)^{-1-x}}$$

$$<n^x(n-1)!\sqrt{e^{\frac{1-x}{n-1}-\frac{2(1-x^2)}{2n+x-1}}}$$

$$<n^x(n-1)!\,e^{\frac{1-x}{2(n-1)}}$$

i.e.,

(11) $$f(n+x)<n^x(n-1)!\,e^{\frac{1}{2(n-1)}},\quad x\in(0,1),\quad n\geqq 2+[A].$$

Further, with

$$a=n+x-1,\quad b=n,\quad c=n+x,\quad d=n+1,\quad x\in(0,1),\quad n\geqq 2+[A],$$

from (9) we have

$$(f(n))^{2-x}(f(n+1))^x<(f(n+x-1))^x(f(n+x))^{2-x},$$

i.e.,

$$(n-1)!\,(n-1)^x<f(n+x),\quad x\in(0,1),\quad n\geqq 2+[A].$$

This last inequality holds also if n is substituted by $n+1$. Therefore we have

(12) $$\frac{n!\,n^x}{n+x}<f(n+x),\quad x\in(0,1),\ n\geqq 2+[A].$$

On account of (8), (11) and (12) we conclude

(13) $$\frac{n!\,n^x}{x(x+1)\cdots(x+n)}<f(x)<\frac{n!\,n^x}{x(x+1)\cdots(x+n)}\cdot\frac{n+x}{n}\,e^{\frac{1}{2(n-1)}}.$$

Since

$$\Gamma(x)=\lim_{n\to+\infty}\frac{n!\,n^x}{x(x+1)\cdots(x+n)},$$

if $n\to+\infty$, the inequalities (13) show that

$$f(x)=\Gamma(x)\quad\text{on}\quad(0,1).$$

It is clear that from the above remarks (see (8)) we have

$$f(x)=\Gamma(x)\quad\text{on}\quad(0,+\infty).$$

Now let k be a natural number, and let us denote

$$F_k(x)=(-1)^{k+1}\ln f(x)$$

where the function $f\colon(0,+\infty)\to\mathbf{R}$ is positive. In a similar way, with the proof of the above theorem, we may establish

Theorem 4. *If* $f\colon(0,+\infty)\to\mathbf{R}$ *is positive on* $(0,1)$ *and satisfies the following conditions*

(i) $f(x+1)=xf(x),\quad x\in(0,+\infty)$,

(ii) F_k *is a convex function of the order* k *on* $(A,+\infty)$, A *being a fixed non-negative number,*

(iii) $f(1)=1$,

then we have

$$f(x)=\Gamma(x)\quad \textit{for}\quad x\in(0,+\infty).$$

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