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493. AN INEQUALITY FOR POLYNOMIALS*

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The purpose of this paper is to give the best constant $K_n(s, \alpha, \beta)$ in the inequality

$$\|f^{(s)}\| \leqq K_n(s, \alpha, \beta)\, \|f\|_2$$

where f is a polynomial of degree n,

$$\|f^{(s)}\| = \max_{x \in [-1, +1]} |f^{(s)}(x)|$$

and

$$\|f\|_2 = \left\{ \int_{-1}^{1} w(x)\, |f(x)|^2 \, dx \right\}^{1/2}$$

with

$$w(x) = (1-x)^{\alpha} (1+x)^{\beta}, \quad \alpha > -1, \quad \beta > -1.$$

For $\alpha = 0$, $\beta = 0$ the constant $K_n(s, 0, 0)$ is due to GILBERT LABELLE in [**1**].

In this paper we use the following notation:

a) $$A_k^{(\alpha, \beta)} = \left\{ \frac{(2k+\alpha+\beta+1)\, k!\, \Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+1}\, \Gamma(k+\alpha+1)\, \Gamma(k+\beta+1)} \right\}^{1/2}.$$

b) $J_k(x) = A_k^{(\alpha, \beta)} \cdot P_k^{(\alpha, \beta)}(x)$, $k = 0, 1, \ldots$, is the orthonormal system of JACOBI polynomials, (α, β fixed).

c) $$C_{k,s}^{(\alpha, \beta)} = \frac{k!\,(2k+\alpha+\beta+1)\, \Gamma(k+\alpha+\beta+s+1)}{\Gamma(k+\alpha+1)\, \Gamma(k+\beta+1)} \binom{k+q}{k-s}.$$

d) $$q = \max(\alpha, \beta).$$

Theorem. *If f is a polynomial of degree n and $q \geqq -\frac{1}{2}$, then holds the inequality*

$$\|f^{(s)}\| \leqq K_n(s, \alpha, \beta)\, \|f\|_2 \tag{1}$$

where the best constant is given by

(2) $K_n(s, \alpha, \beta) =$

$$= \left\{ \frac{s!}{2^{2s+\alpha+\beta+1}} \sum_{k=s}^{n} \frac{k!\,(2k+\alpha+\beta+1)\, \Gamma(k+\alpha+\beta+s+1)}{\Gamma(k+\alpha+1)\, \Gamma(k+\beta+1)} \binom{k+\alpha+\beta+s}{s} \binom{k+q}{k-s}^2 \right\}^{1/2}.$$

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The equality in (1) *holds for the polynomials*

$$(3)\qquad f^*(x, \alpha, \beta) = C \cdot \sum_{k=s}^{n} C_{k,s}^{(\alpha,\beta)} \cdot P_k^{(\alpha,\beta)}(x),$$

where C *is a constant and* $P_k^{(\alpha,\beta)}$, $k=0, 1, \ldots,$ *are the Jacobi polynomials.*

Proof. Taking into account that

$$\frac{\mathrm{d}}{\mathrm{d}x}\{P_k^{(\alpha,\beta)}(x)\} = \frac{k+\alpha+\beta+1}{2} P_{k-1}^{(\alpha+1,\beta+1)}(x)$$

(see [2]), for $k=s, s+1, \ldots, n$ we have

$$|J_k^{(s)}(x)|^2 = A_{k,s} |P_{k-s}^{(\alpha+s,\beta+s)}(x)|^2$$

with

$$A_{k,s} = \frac{s!\,k!\,(2k+\alpha+\beta+1)\,\Gamma(k+\alpha+\beta+s+1)}{2^{2s+\alpha+\beta+1}\,\Gamma(k+\alpha+1)\,\Gamma(k+\beta+1)} \binom{k+\alpha+\beta+s}{s}.$$

Let us suppose that $\max(\alpha, \beta) \geqq -\frac{1}{2}$. It is known (see [2], theorem 7.32.1) that

$$\max_{x\in[-1,+1]} |P_{k-s}^{(\alpha+s,\beta+s)}(x)| = |P_{k-s}^{(\alpha+s,\beta+s)}(\bar{x})| = \binom{k+q}{k-s} \quad .$$

where the maximum is attained by $\bar{x} = -1$ $(q=\beta)$ or, in the case $q=\alpha$ by $\bar{x}=+1$.

Therefore on $[-1, +1]$ we have

$$(4)\qquad \sum_{k=s}^{n} |J_k^{(s)}(t)|^2 \leqq \sum_{k=s}^{n} A_{k,s} \binom{k+q}{k-s}^2$$

with equality for

$$(5)\qquad t=-1 \quad (\beta \geqq \alpha) \text{ or } t=1 \quad (\alpha \geqq \beta).$$

Now let $f(x) = \sum_{k=0}^{n} a_k x^k$; then

$$f(x) = \sum_{k=0}^{n} f_k \cdot J_k(x)$$

where

$$f_k = \int_{-1}^{1} w(x) f(x) J_k(x)\,\mathrm{d}x, \quad k=0, 1, \ldots, n,$$

and it is known that

$$\|f\|_2 = \left\{\sum_{k=0}^{n} |f_k|^2\right\}^{1/2}.$$

Thus for an arbitrary t holds the inequality

$$(6)\qquad |f^{(s)}(t)| \leqq \left\{\sum_{k=s}^{n} |J_k^{(s)}(t)|^2\right\}^{1/2} \cdot \|f\|_2$$

with equality for

$$f(x)=C\cdot\sum_{k=s}^{n}J_k^{(s)}(t)\cdot J_k(x). \tag{7}$$

In view of (4)–(6), we have

$$\|f^{(s)}\|\leq K_n(s,\alpha,\beta)\,\|f\|_2$$

where $K_n(s,\alpha,\beta)$ is given by (2).

The conditions (5)–(7) furnish the extremal polynomials

$$f^*(x,\alpha,\beta)=C\cdot\sum_{k=s}^{n}C_{k,s}^{(\alpha,\beta)}\cdot P_k^{(\alpha,\beta)}(x)$$

and this completes the proof of the above theorem.

Corollary. *If we denote by $q'=\min(\alpha,\beta)$, then for every polynomial of degree n holds the inequality*

$$\|f\|\leq\|f\|_2\cdot\sqrt{\frac{\Gamma(n+\alpha+\beta+2)}{2^{\alpha+\beta+1}\,\Gamma(q+1)\,\Gamma(n+q'+1)}\binom{n+q+1}{n}}.$$

The equality is valid for the polynomials

$$f^*(x)=C\cdot\sum_{k=0}^{n}\frac{(2k+\alpha+\beta+1)\,\Gamma(k+\alpha+\beta+1)}{\Gamma(k+q'+1)}P_k^{(\alpha,\beta)}(x).$$

Proof. We have

$$K_n(0,\alpha,\beta)=\left\{\sum_{k=0}^{n}|J_k(\bar{x})|^2\right\}^{1/2},\quad |\bar{x}|=1. \tag{8}$$

Moreover

$$J_k(x)=(M_k x+N_k)\cdot J_{k-1}(x)+D_k\cdot J_{k-2}(x),\quad k=2,3,\ldots,$$

where N_k, D_k are real numbers and

$$M_k=\frac{2k+\alpha+\beta}{2}\left[\frac{(2k+\alpha+\beta+1)(2k+\alpha+\beta-1)}{k(k+\alpha+\beta)(k+\alpha)(k+\beta)}\right]^{1/2}.$$

At the same time it is known that (see [**2**])

$$\sum_{k=0}^{n}|J_k(t)|^2=\frac{1}{M_{n+1}}\left[J'_{n+1}(t)J_n(t)-J'_n(t)\,J_{n+1}(t)\right]. \tag{9}$$

Because

$$J'_{n+1}(\bar{x})=\sqrt{A_{n+1,1}}\binom{n+q+1}{n},\qquad J_n(\bar{x})=A_n^{(\alpha,\beta)}\binom{n+q}{n}$$

the equalities (8) and (9), with $t=\bar{x}$, give us the form of the best constant.

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