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466. SOME REMARKS ON THE PAPER "A NOTE ON AN INEQUALITY" OF V. K. LIM*

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Professor D. S. Mitrinović has drawn our attention to paper [2] and its review [3] and suggested that we examine whether the results of the mentioned paper can be proved by a direct application of known properties of convex functions.

1. In paper [2] V. K. Lim has proved the following two theorems:

Theorem A. *Let* a *and* b *be two non-negative real numbers and* $a+b\leqq c$. *Then*

$$a^r+(b+c)^r\geqq(a+b)^r+c^r$$

for $r\geqq 1$.

Inequality is reversed if $0<r<1$, *and equality holds if either* $b=0$ *or* $r=1$.

Theorem B. *Let* $Q_1, Q_2, \ldots, Q_m$ *be non-negative numbers,* $r>0$ *and*

$$Q\geqq\max(Q_1, Q_2, \ldots, Q_m).$$

(i) *If* $sQ=\sum_{i=1}^{m}Q_i$, *for some real* s, *then*

$$sQ^r\geqq\sum_{i=1}^{m}Q_i^r\quad(r\geqq 1),\qquad sQ^r\leqq\sum_{i=1}^{m}Q_i^r\quad(0<r<1).\tag{2}$$

Equality holds if one of the following holds:

(a) $r=1$,

(b) $Q_i=0$ *for* $1\leqq i\leqq m$,

(c) s *is an integer* $(1\leqq s\leqq m)$ *such that* $Q_i=Q$ *for* $1\leqq i\leqq s$ *and* $Q_i=0$ *when* $i>s$.

(ii) *If* $sQ+Q_0=\sum_{i=1}^{m}Q_i$ *for some real* s, Q_0 *where* $0<Q_0<Q$, *and either* $s=0$ *or if* $s\neq 0$, $sQ\geqq Q_i$ *for* $1\leqq i\leqq m$, *then*

$$sQ^r+Q_0^r\geqq\sum_{i=1}^{m}Q_i^r\qquad(r\geqq 1).\tag{3}$$

* Presented November 13, 1973 by P. M. Vasić.

Equality holds if one of the following occurs:

(a) $r=1$,

(b) $s=0$, $Q_0=Q_i$ *for some* i *and* $Q_j=0$ *for* $j\neq i$,

(c) s *is an integer such that there are* s Q_i*'s equal to* Q, *one* $Q_i=Q_0$ *and the rest of* Q_i*'s equal to zero.*

As it was noticed by H. T. CROFT [3] the conclusion (ii) of theorem B (i.e. inequality (3)) is not correct for arbitrary real number s. Counter example from [3] is

$$Q=Q_1=Q_2=1,\quad Q_0=0{,}5,\quad s=1{,}5,\quad m=r=2.$$

On the other hand theorems A and B are immediate consequences of some, well known elementary properties of convex functions, as it will be shown in a much simpler way than in the paper [2].

2. Namely we will prove the following two theorems:

Theorem 1. *Let* a *and* b *be nonnegative real numbers and let* $a+b\leqq c$. *Then for every convex functions* $x\mapsto f(x)$ *defined for all* $x\geqq 0$ *the following inequality holds*

$$f(a)+f(b+c)\geqq f(a+b)+f(c). \tag{4}$$

If the function f *is concave the above inequality is reversed.*

In the case when $f(t)=t^r$ where $t\geqq 0$ and where $r\geqq 1$ or $0<r<1$ Theorem 1 reduces to the Theorem A.

Proof. From the assumption of our theorem it follows that $a\leqq b\leqq c$ or $b\leqq a\leqq c$. If we introduce the notations $x_1=b+c$, $x_2=a$, $y_1=c$, $y_2=a+b$ then we have $x_1\geqq x_2$, $y_1\geqq y_2$, $x_1+x_2=y_1+y_2$, i.e. the vector y is majorized by the vector x, where $x=(x_1, x_2)$, $y=(y_1, y_2)$ (which is, as usual denoted by $x\succ y$, see [1] page 162). Therefrom and from well known theorem of HARDY—LITTLEWOOD—PÓLYA (see [1] page 164 theorem 1) it follows that for every convex function f the following inequality holds

$$f(x_1)+f(x_2)\geqq f(y_1)+f(y_2)$$

in virtue of what the theorem is proved.

Theorem 2. *Let us suppose that the numbers* $Q_1, \dots, Q_m$ $(m\in N)$ *are nonnegative and that* $Q\geqq \max(Q_1, \dots, Q_m)$. *If the real number* s *is such that* $sQ=\sum_{i=1}^{m} Q_i$ *then the following inequality holds*

$$sf(Q)\geqq \sum_{i=1}^{m} f(Q_i) \tag{5}$$

for every convex function $x\mapsto f(x)$, *defined for all* $x\geqq 0$, *for which we have* $f(0)=0$.

Proof. If $Q=0$, then $Q_i=0$ $(i=1, \dots, m)$ so that (5) is reduced to equality, in virtue of $f(0)=0$. Further we can suppose that $Q>0$. Then as for every convex function the inequality $f(ax)\leqq af(x)$ holds for every $x\geqq 0$ and $0\leqq a\leqq 1$ we have

$$f(Q_i)=f\left(\frac{Q_i}{Q}Q\right)\leqq\frac{Q_i}{Q}f(Q)\qquad(i=1,\dots,m)\tag{6}$$

in virtue of $0\leqq\frac{Q_i}{Q}\leqq 1$ $(i=1,\dots,m)$. Therefrom adding inequalities (6) from $i=1$ to $i=m$ we get

$$\sum_{i=1}^{m}f(Q_i)\leqq\frac{f(Q)}{Q}\sum_{i=1}^{m}Q_i=sf(Q),\tag{7}$$

which proves the theorem.

Theorem 2.1. *Let us suppose that the numbers $Q_1,\dots,Q_m$ $(m\in N)$ are nonnegative and let $Q\geqq\max(Q_1,\dots,Q_m)$. If n is natural number such that*

$$nQ+Q_0=\sum_{i=1}^{m}Q_i,$$

where $0<Q_0<Q$, then the following inequality

$$nf(Q)+f(Q_0)\geqq\sum_{i=1}^{m}f(Q_i)\tag{8}$$

is valid, for every convex function $x\mapsto f(x)$, defined for all $x\geqq 0$ and for which we have $f(0)=0$.

Proof. In (8) the equality is valid if

(a) $Q=0$ (then $Q_0=Q_i=0$ $(i=1,\dots,m)$),

(b) $n=0$, $Q_0=Q_{i_0}$ and $Q_j=0$ for $j\neq i_0$.

If we suppose that $Q>0$ then we have

$$nQ<nQ+Q_0=\sum_{i=1}^{m}Q_i\leqq mQ$$

wherefrom it follows that $n<m$.

Let us introduce the following notations

$$x_i=Q\ (i=1,\dots,n)\ \text{and}\ x_{n+1}=Q_0\ \text{if}\ m=n+1$$

or

$$x_i=Q\ (i=1,\dots,n),\ x_{n+1}=Q_0,\ x_i=0\quad(i=n+2,\dots,m)$$

if $m\geqq n+2$, and $y_i=Q_i$ $(i=1,\dots,m)$. Besides, we can suppose that $Q_i\geqq Q_{i+1}$ without loss of generality. So, we have

$$x_1\geqq x_2\geqq\dots\geqq x_m\quad\text{and}\quad y_1\geqq y_2\geqq\dots\geqq y_m,$$

$$\sum_{i=1}^{k}x_i\geqq\sum_{i=1}^{k}y_i\qquad(k=1,\dots,m-1),\qquad\sum_{i=1}^{m}x_i=\sum_{i=1}^{m}y_i,$$

i.e. $x\succ y$ where $x=(x_1,\dots,x_m)$, $y=(y_1,\dots,y_m)$. In virtue of the cited theorem of HARDY—LITTLEWOOD—PÓLYA we can conclude that

$$\sum_{i=1}^{m}f(x_i)\geqq\sum_{i=1}^{m}f(y_i).$$

Since we have

$$\sum_{i=1}^{m} f(x_i) = nf(Q) + f(Q_0), \quad \sum_{i=1}^{m} f(y_i) = \sum_{i=1}^{m} f(Q_i)$$

the validity of inequality (8) follows.

REMARK. In this paper we have taken convex functions to mean continuous and JENSEN convex functions, as defined in [1].

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