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461. HISTORY, VARIATIONS AND GENERALISATIONS OF THE ČEBYŠEV INEQUALITY AND THE QUESTION OF SOME PRIORITIES*

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In CH. HERMITE's letter of 27 January 1887 addressed to T. J. STIELTJES there is a part reading:

„... *Il est bien difficile d'avoir tout ce qui se publie à notre époque si féconde, présent à l'esprit, et cette difficulté s'augmente pour moi de mon ignorance de l'allemand, ce qui vous explique pourquoi j'ai attribué à M. Markoff, qui a écrit son article en français, ce qu'avait déjà fait M. M. Bruns dans le Journal de Borchardt. Mais ce me sera un plaisir quand je ferai pour l'impression une rédaction plus correcte de mon Cours lithographié d'y donner place à votre Travail qui excite extrêmement ma curiosité d'après le peu que vous m'en dites...*"

É. PICARD in his letter addressed to the director of the journal Rendiconti del Circolo matematico di Palermo, of 28 June 1913, says:

„... *Cette remarque, comme vous pensez bien, n'a pas pour objet une ridicule réclamation de priorité. Je veux seulement remarquer combien il est difficile aujourd'hui de faire une bibliographie ayant quelque valeur historique. Il serait peut-être exact de dire que la moitié des attributions sont fausses, et que bien souvent on ne cite pas le premier inventeur. L'histoire des sciences deviendra de plus en plus difficile à écrire, je ne compte guère, pour remonter le courant, sur les encyclopédies où l'historien risque de se noyer dans un flot de citations au milieu desquelles disparaît celui qui a eu la première idée. Vous rendez, cher Ami, un grand service dans les Rendiconti en faisant réviser et compléter souvent les citations des auteurs, en partie peut-être responsables de cet état des choses par le peu soin qu'ils apportent aux indications bibliographiques. Soyez d'ailleurs assuré que je me dis en ce moment que celui qui est sans péché lui jette la première pierre.*"

G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA in their book *Inequalities*, in the Preface to the first edition of 1934, say:

„... *Historical and bibliographical questions are particularly troublesome in a subject like this, which has applications in every part of mathematics but has never been developed systematically.*

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* Presented January 25, 1974 by P. R. BEESACK.

It is often really difficult to trace the origin of a familiar inequality. It is quite likely to occur first as an auxiliary proposition, often without explicit statement, in a memoir on geometry or astronomy; it may have been rediscovered, many years later, by half a dozen different authors; and no accessible statement of it may be quite complete... We have done our best to be accurate and have given all references we can, but we have never undertaken systematic bibliographical research. We follow the common practice, when a particular inequality is habitually associated with a particular mathematician's name; we speak of the inequalities of Schwarz, Hölder and Jensen, though all these inequalities can be traced further back; ..."

The authors of this paper have studied, for considerable time, the literature devoted to ČEBYŠEV's inequality. Many articles have been written on that subject and some of them are not easy of access.

This paper offers a history of ČEBYŠEV's inequality and related inequalities. Certain facts which contest some priorities, accepted by the main monographs on inequalities, are brought to light.

Incorrect quotations keep on being uncritically transferred from book to book, from paper to paper. This is practised even by some very distinguished mathematicians. Hence, there are many „attributions fausses" regarding ČEBYŠEV's inequality.

Some results on ČEBYŠEV's inequality have been rediscovered several times. There are also a considerable number of papers which offer apparent generalisations as new results.

0. Let $f, g \colon [a, b] \to \mathbf{R}$ be integrable functions, both increasing or both decreasing. Furthermore, let $p \colon [a, b] \to \mathbf{R}_0^+$ be an integrable function. Then

$$(0.1) \qquad \int_a^b p(x)\,dx \int_a^b p(x) f(x) g(x)\,dx \geqq \int_a^b p(x) f(x)\,dx \int_a^b p(x) g(x)\,dx.$$

If one of the functions f or g is increasing and the other decreasing, then the sign of inequality is reversed in (0.1).

Inequality (0.1) is known in the literature as ČEBYŠEV's inequality. The following special cases of (0.1)

$$(0.2) \qquad \int_a^b f(x) g(x)\,dx \geqq \frac{1}{b-a} \int_a^b f(x)\,dx \int_a^b g(x)\,dx.$$

and

$$(0.3) \qquad \int_0^1 f(x) g(x)\,dx \geqq \int_0^1 f(x)\,dx \int_0^1 g(x)\,dx$$

are also called ČEBYŠEV's inequalities.

For each of the above inequalities there exists a corresponding discrete analogue. So, for example, if $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are two nondecreasing (or nonincreasing) sequences and if $p = (p_1, \ldots, p_n)$ is a nonnegative sequence, then

$$(0.4) \qquad \sum_{i=1}^n p_i \sum_{i=1}^n p_i a_i b_i \geqq \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i,$$

with equality if and only if $a = b$.

For $p_1 = \cdots = p_n = 1$, we obtain the following special case of (0.4)

$$\sum_{i=1}^{n} a_i b_i \geqq \frac{1}{n} \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i, \tag{0.5}$$

also called ČEBYŠEV's inequality.

1. It appears that the first results regarding the inequalities, lately referred to as ČEBYŠEV's inequalities, have been obtained by LAPLACE (1749—1827). Namely in the book by G. CHRYSTAL ([**1**], p. 50) the credit for the following result has been given to LAPLACE:

If $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are two positive decreasing (or increasing) sequences, then

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i^2 y_i > \sum_{i=1}^{n} x_i y_i \sum_{i=1}^{n} x_i^2. \tag{1.1}$$

This result is a special case of inequality (0.4) for $p_i = a_i = x_i$ and $b_i = y_i$.

2. In the paper [**2**] of A. WINCKLER from 1866 the following result is proved:

If f and g are bounded and positive functions, one of which is nondecreasing and the other nonincreasing starting from $x=0$, then

$$\int_0^x f(x)\, g(x)\, \mathrm{d}x < \frac{1}{x} \int_0^x f(x)\, \mathrm{d}x \int_0^x g(x)\, \mathrm{d}x. \tag{2.1}$$

The proof of (2.1) given by A. WINCKLER is based on geometric reasoning and is rather complicated. We give here also the authentic formulation of his result, which is, in fact, ČEBYŠEV's inequality:

Wenn $\varphi(x)$ und $\vartheta(x)$ zwei endlich und positiv bleibende Functionen von x sind, wovon die eine von $x=0$ an nie abnimmt, die andere von $x=0$ an nie wächst, so ist:

$$\int_0^x \varphi(x)\, \vartheta(x)\, \mathrm{d}x < \frac{1}{x} \left[\int_0^x \varphi(x)\, \mathrm{d}x\right] \left[\int_0^x \vartheta(x)\, \mathrm{d}x\right].$$

3. P. L. ČEBYŠEV has given in [**3**] the following theorem without a proof:

If p, f, g are real functions such that $x \mapsto p(x) f(x) g(x)$ is integrable on $[a, b]$ and $p(x) > 0$ for $x \in [a, b]$, then

$$\int_a^b p(x) f(x) g(x)\, \mathrm{d}x = \frac{\int_a^b p(x) f(x) h_0(x)\, \mathrm{d}x \int_a^b p(x) g(x) h_0(x)\, \mathrm{d}x}{\int_a^b p(x) h_0(x)^2\, \mathrm{d}x} + \cdots$$

$$+ \frac{\int_a^b p(x) f(x) h_{n-1}(x)\, \mathrm{d}x \int_a^b p(x) g(x) h_{n-1}(x)\, \mathrm{d}x}{\int_a^b p(x) h_{n-1}(x)^2\, \mathrm{d}x} + R_n,$$

where $h_0, h_1, \ldots, h_{n-1}$ are denominators of the continued fractions obtained by developing $\int\limits_a^b \frac{p(z)}{x-z}\,\mathrm{d}z$ into continued fractions. In addition,

$$1^\circ \quad |R_n| \leqq \frac{\int\limits_a^b p(x)\, h_n(x)^2\,\mathrm{d}x}{\left(\frac{\mathrm{d}^n h_n(x)}{\mathrm{d}x^n}\right)^2} \max_{a\leqq x\leqq b} \left|\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right| \max_{a\leqq x\leqq b} \left|\frac{\mathrm{d}^n g(x)}{\mathrm{d}x^n}\right|,$$

2° If derivatives $\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}$ and $\frac{\mathrm{d}^n g(x)}{\mathrm{d}x^n}$ do not change the sign on $[a, b]$ then

$$\operatorname{sgn} R_n = \operatorname{sgn}\left(\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\,\frac{\mathrm{d}^n g(x)}{\mathrm{d}x^n}\right).$$

For $n=1$ the above theorem implies:

If p, f, g are integrable functions and $p(x)>0$ on $[a, b]$ and if $\operatorname{sgn}\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \operatorname{sgn}\frac{\mathrm{d}g(x)}{\mathrm{d}x}$ on $[a, b]$, then inequality (0.1) is valid.

Čebyšev had submitted this paper for publication at the beginning of 1883. However, as the paper caused great interest, the Editorial Committee published it in the last issue of the volume for 1882.

This is the remark in paper **[8]** of the Editorial Committee.

Somewhat later (in 1883) Čebyšev published in [7] the proofs of his results, which were in no way altered.

4. Ch. Hermite introduced in his lectures at the Sorbonne (Paris) in 1881/82 (see [4]) the inequality (0.3). The comment: „... un résultat bien remarquable dont je dois la communication à M. Tchebichew.“ was later interpreted as if Čebyšev had not proved this inequality; this as we will see is not correct. Hermite also introduced in his lectures the proof given by Picard from which it can be understood that Čebyšev had presented to him the inequality without any proof. Picard's proof, based on mechanical facts, textually is:

Soit u et v deux fonctions de x, qui entre les valeurs $x=0$, et $x=1$ soient positives, et varient l'une et l'autre dans le même sens, de sorte qu'elles soient continuellement croissantes ou continuellement décroissantes on aura l'inégalité:

$$\int\limits_0^1 uv\,\mathrm{d}x > \int\limits_0^1 u\,\mathrm{d}x \int\limits_0^1 v\,\mathrm{d}x.$$

Mais en supposant que l'une des fonctions soit croissante, et l'autre décroissante on devra prendre au contraire:

$$\int\limits_0^1 uv\,\mathrm{d}x < \int\limits_0^1 u\,\mathrm{d}x \int\limits_0^1 v\,\mathrm{d}x.$$

Ce théorème de l'illustre analyste a été démontré comme il suit par M. Picard:

Proposons-nous par exemple d'établir que

$$\int_0^1 u\,dx \int_0^1 v\,dx < \int_0^1 uv\,dx,$$

u et v étant deux fonctions de x, qui sont positives et vont en décroissant quand x varie de zéro à l'unité. Si l'on partage cet intervalle en n parties égales et que l'on pose $dx = \frac{1}{n}$, les valeurs de u et v étant $u_1, u_2, \ldots, u_n$ et $v_1, v_2, \ldots, v_n$ pour $x = \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$, l'inégalité précédente sera établie si on montre que

$$(u_1+u_2+\cdots+u_n)(v_1+v_2+\cdots+v_n) < n(u_1v_1+u_2v_2+\cdots+u_nv_n)$$

en supposant, d'après ce qui a été dit de u et v, que

$$u_1 > u_2 > \cdots > u_n \quad \text{et} \quad v_1 > v_2 > \cdots > v_n.$$

Nous écrirons l'inégalité précédente sous la forme

$$\frac{u_1+u_2+\cdots+u_n}{n} < \frac{u_1v_1+u_2v_2+\cdots+u_nv_n}{v_1+v_2+\cdots+v_n}.$$

Or portons en abscisse sur une droite les longueurs $u_1, u_2, \ldots, u_n$ ce qui nous donnera les points $A_1, A_2, \ldots, A_n$ le premier membre de l'inégalité précédente représente le centre de gravité des points $A_1, A_2, \ldots, A_n$ quand on leur donne la même masse et nous supposerons que cette masse soit v_n. Le second membre représente le centre de gravité des mêmes points quand on leur donne respectivement pour masse $v_1, v_2, \ldots, v_n$. Mais décomposons ce dernier système de la manière suivante: soit tout d'abord

le système des points	$A_n, A_{n-1}, \ldots, A_1$	de masse v_n,
puis le système	$A_{n-1}, \ldots, A_1$	de masse $v_{n-1}-v_n$,
	$A_{n-2}, \ldots, A_1$	de masse $v_{n-2}-v_{n-1}$,
	$\vdots$	
	A_2, A_1	de masse v_2-v_3,
	A_1	de masse v_1-v_2.

On voit de suite que ces divers systèmes réunis donnent le système de points $A_1, A_2, \ldots, A_n$ avec les masses $v_1, v_2, \ldots, v_n$. Or les centres de gravités de ces divers systèmes

$$\frac{u_1+u_2+\cdots+u_n}{n} < \frac{u_1+\cdots+u_{n-1}}{n-1} < \frac{u_1+\cdots+u_{n-2}}{n-2} < \cdots < u_1.$$

Or étant donné un système de points sur une droite, leur centre de gravité est évidemment situé entre les deux extrêmes, c'est-à-dire ici outre

$$\frac{u_1+u_2+\cdots+u_n}{n} \quad \text{et} \quad u_1.$$

On a donc:

$$\frac{u_1 v_1 + \cdots + u_n v_n}{v_1 + v_2 + \cdots + v_n} < \frac{u_1 + u_2 + \cdots + u_n}{n}.$$

Les autres cas de la proposition de M. TCHEBICHEW se démontreraient de la même manière.

Note that PICARD as well as WINCKLER introduced an additional assumption concerning positivity of the functions which appear in inequality (0.3). This is, in view of the methods which they have used, quite natural.

As we see PICARD proved the inequality (0.5) at the same time.

5. In his letter to N. V. BUGAEV (see [**5**]) A. KORKINE started from the identity

$$\frac{1}{n} \sum_{i=1}^{n} x_i y_i = \left(\frac{1}{n} \sum_{i=1}^{n} x_i\right)\left(\frac{1}{n} \sum_{i=1}^{n} y_i\right) + \frac{1}{n^2} \sum_{i<j} (x_i - x_j)(y_i - y_j). \tag{5.1}$$

Putting here $x_i = f\left(\frac{i-1}{n}\right)$ and $y_i = g\left(\frac{i-1}{n}\right)$, and leting $n \to +\infty$, KORKINE obtained inequality (0.3).

We observe that from (5.1) one immediately obtains the discrete case of inequality (0.5).

The same results can be found in KORKINE's letter to HERMITE (see [**6**]).

6. In [7] ČEBYŠEV published the proofs of his results from paper [**2**]. As can be seen, the opinion that ČEBYŠEV gave inequalities which bear his name without proof is not correct, though it can often be found in the literature. ČEBYŠEV not only proved special cases (0.2) and (0.3), but also the most general inequality (0.1), though under stronger conditions. ČEBYŠEV introduced suppositions on the derivatives of f and g, which he had to do because of the method he used.

7. ČEBYŠEV's younger collaborator K. POSSÉ stated his own proofs of ČEBYŠEV's result from paper [3] at the session of Karkov's Mathematical Society on 5th May 1883. These proofs were published in Russian [**8**] and also in the same year in French [**9**].

8. V. G. IMŠENECKIĬ [**10**] has applied the method of KORKINE in order to get some inequalities analogous to ČEBYŠEV's ones.

9. By using an identity analogous to (5.1), where integrals appear instead of finite sums, K. A. ANDRÉIEF [**11**] considered the following identity:

$$\int_a^b F_1(x) F_2(x)\, dx \int_a^b G_1(x) G_2(x)\, dx - \int_a^b F_1(x) G_2(x)\, dx \int_a^b F_2(x) G_1(x)\, dx$$

$$= \frac{1}{2} \int_a^b \int_a^b \big(F_1(x) G_1(y) - F_1(y) G_1(x)\big)\big(F_2(x) G_2(y) - F_2(y) G_2(x)\big)\, dx\, dy.$$

From this, for $F_1(x)=f(x)$, $F_2(x)=g(x)$ and $G_1(x)=G_2(x)=1$, we obtain

$$(9.1)\qquad (b-a)\int_a^b f(x)\,g(x)\,\mathrm{d}x-\int_a^b f(x)\,\mathrm{d}x\int_a^b g(x)\,\mathrm{d}x$$

$$=\frac{1}{2}\int_a^b\int_a^b \big(f(x)-f(y)\big)\big(g(x)-g(y)\big)\,\mathrm{d}x\,\mathrm{d}y,$$

which implies inequality (0.2).

Also in **[11]** ANDRÉIEF generalized inequality (0.3) to n functions, namely he proved the following theorem:

Let each among the functions $f_1, \ldots, f_n$, and among their derivatives $f_1', \ldots, f_n'$ not change sign on [0, 1] and let, further, all quotients

$$\frac{f_1'(x)}{f_1(x)}, \ldots, \frac{f_n'(x)}{f_n(x)}$$

have the same sign on [0, 1]. Then, if among $f_1, \ldots, f_n$ there exists an even number of positive functions on [0, 1], we have

$$(9.2)\qquad \int_0^1 f_1(x)\cdots f_n(x)\,\mathrm{d}x \geqq \int_0^1 f_1(x)\,\mathrm{d}x\cdots\int_0^1 f_n(x)\,\mathrm{d}x.$$

If the number of the positive functions in question is odd, the inequality (9.2) is reversed.

10. In 1883 C. ANDRÉIEF published another paper **[12]** relevant to ČEBYŠEV's inequality. Primarily, he showed that for arbitrary continuous functions $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ and $p\,(p(x)>0)$ the following identity is valid

$$(10.1)\qquad \begin{vmatrix} \int_a^b p(x)f_1(x)g_1(x)\,\mathrm{d}x & \ldots & \int_a^b p(x)f_1(x)g_n(x)\,\mathrm{d}x \\ \vdots & & \\ \int_a^b p(x)f_n(x)g_1(x)\,\mathrm{d}x & & \int_a^b p(x)f_n(x)g_n(x)\,\mathrm{d}x \end{vmatrix}$$

$$=\frac{1}{n!}\int_a^b\cdots\int_a^b \begin{vmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & & \\ f_n(x_1) & & f_n(x_n) \end{vmatrix} \begin{vmatrix} g_1(x_1) & \cdots & g_1(x_n) \\ \vdots & & \\ g_n(x_1) & & g_n(x_n) \end{vmatrix} p(x_1)\cdots p(x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n.$$

From this identity various inequalities can be obtained, as for example those of ČEBYŠEV, CAUCHY, etc. Simultaneously, ANDRÉIEF generalized ČEBYŠEV's and POSSÉ's results on the development of the function $\int_a^b p(x)f(x)\,g(x)\,\mathrm{d}x$ into a series.

11. A. WINCKLER published a paper [**13**] in 1884 where he gave two proofs of inequality (0.2). In this paper on p. 528 he says: Den Beweis sowie den Satz lernte ich aus dem Cours professé à la faculté des sciences par M. HERMITE, (1881—82), second tirage, Paris 1883, kennen.

WINCKLER therefore claims that he did not know of other papers on ČEBYŠEV's inequality, and hence it is not surprising that his proofs are, in fact, identical with the proofs of KORKINE [**5**], [**6**] and ANDRÉIEF [**11**]. It is, however, interesting that he makes no mention of his own paper [**2**] in which ČEBYŠEV's inequality is proved under the conditions which are the same as PICARD's, whose proof WINCKLER claims to know, and no other.

12. In 1885 F. FRANKLIN [**14**] proved inequality (0.2) using identity (9.1) (i.e. using the same identity as ANDRÉIEF). In the same paper he gave a proof of inequality (0.5) using an identity already used by KORKINE [**5**], [**6**] in the proof of the same inequality. HERMITE was also unaware of papers of KORKINE and ANDRÉIEF, since in 1891 he included ANDRÉIEF's proof in his lectures [**18**], but ascribed it to FRANKLIN. Many other authors also ascribed the proof of ČEBYŠEV's inequality which uses identity (9.1) to FRANKLIN (see, for example, [**19**], [**24**], [**25**], [**26**], [**37**], [**40**], [**41**], [**45**]).

13. In 1888 one more proof of ČEBYŠEV's inequality appeared which is identical with KORKINE's. That was paper [**15**] due to student D'ARONE.

14. JENSEN published in 1888 paper [**16**] which contains no references. He has obviously generalised inequality (0.5). JENSEN's proof reads:

Let $u_1, \ldots, u_n$; $v_1, \ldots, v_n$; $w_1, \ldots, w_n$ be sequences of positive numbers such that

$$u_1 \geqq u_2 \geqq u_3 \geqq \cdots \geqq u_n, \quad \frac{v_1}{w_1} \geqq \frac{v_2}{w_2} \geqq \cdots \geqq \frac{v_n}{w_n}.$$

Let us put

$$s_n = v_1 + \cdots + v_n, \quad S_n = w_1 + \cdots + w_n.$$

Since $w_m v_n \geqq v_m w_n$ $(m > n)$, we have $S_n s_1 \geqq s_n S_1$, $S_n s_2 \geqq s_n S_2$, .. , $S_n s_{n-1} \geqq s_n S_{n-1}$, so that

$$\frac{\sum\limits_{i=1}^{n} u_i v_i}{\sum\limits_{i=1}^{n} v_i} = \frac{\sum\limits_{i=1}^{n} s_i (u_i - u_{i+1}) + s_n u_n}{s_n} \geqq \frac{\sum\limits_{i=1}^{n} S_i (u_i - u_{i+1}) + S_n u_n}{S_n} = \frac{\sum\limits_{i=1}^{n} u_i w_i}{\sum\limits_{i=1}^{n} w_i}.$$

15. L. GEGENBAUER gave in [**17**], without a proof, a result wherefrom, as a special case ČEBYŠEV's and some other inequalities are obtained. This result textually reads:

Sind die $n^2 (n^{m-2} + 1)$ integrirbaren Functionen

$$\varphi_{i_1, i_2, \ldots, i_m}(x), \ \psi_{i_1, i_2}(x) \qquad (i_1, i_2, \ldots, i_m = 1, 2, \ldots, n)$$

so beschaffen, dass für alle dem Intervalle $a \ldots b$ angehörigen Werthsysteme $x_1, \ldots, x_n$ das Product

$$|\varphi_{i_1, i_2, \ldots, i_{m-1} i_m}(x_{i_m})| \, |\psi_{i, k}(x_k)| \ (i, k, i_1, \ldots, i_{m-1}, i_m = 1, 2, \ldots, n)$$

wo bei negativem m i_m nicht an der Stelle der festen Indices steht, sein Vorzeichen nicht ändert, so hat die über alle Permutationen $\lambda_1, \lambda_2, \ldots, \lambda_n$ der ganzen Zahlen 1, 2, ..., n ausgedehnte Summe

$$\sum_{\lambda_1, \ldots, \lambda_n} \left| \int_a^b \varphi_{i_1, \ldots, i_{m-1}, \lambda_k}(x)\, \psi_{k, \lambda_k}(x)\, dx \right|_{(i_i, \ldots, i_{m-1}, k=1, \ldots, n)}$$

das Vorzeichen dieses Productes.

16. HERMITE, in the fourth edition of his Course **[18]**, inserted the proof of inequality (0.2) by means of identity (9.1), but ascribed it to FRANKLIN.

17. STIELTJES is his letter of 13th February 1891 to HERMITE (see **[19]**) has given the following proof of inequality (0.2):

Permettez-moi de faire une petite remarque sur la proposition de M. TCHEBYCHEFF concernant le signe de l'expression

$$\mathscr{L} = (a' - a) \int_a^{a'} \varphi(x)\, \psi(x)\, dx - \int_a^{a'} \varphi(x)\, dx \times \int_a^{a'} \psi(x)\, dx.$$

A cause de

$$\int_a^{a'} \varphi(x)\, dx = (a' - a)\, \varphi(u),$$

c'est-à-dire

$$(1) \qquad \int_a^{a'} [\varphi(x) - \varphi(u)]\, dx = 0,$$

on a

$$\mathscr{L} = (a' - a) \int_a^{a'} \psi(x)\, [\varphi(x) - \varphi(u)]\, dx$$

ou encore

$$(2) \qquad \mathscr{L} = (a' - a) \int_a^{a'} [\psi(x) - \psi(u)]\, [\varphi(x) - \varphi(u)]\, dx,$$

d'où l'on conclut la proposition de M. TCHEBYCHEFF. On peut dire aussi: L'expression $\mathscr{L}$ ne change pas en remplaçant $\varphi(x)$ par $\varphi(x) + C$, $\psi(x)$ par $\psi(x) + C'$. Donc on peut remplacer $\varphi(x)$ par $\varphi(x) - \varphi(u)$, $\psi(x)$ par $\psi(x) - \psi(u)$. Si l'on prend soin de déterminer u par la relation (1) on obtient ainsi la formule (2).

In his reply to this STIELTJES' letter dated between 16 and 19 February 1891 (which was later lost), HERMITE had some objections to the aforementioned proof. The nature of those objections can be seen from STIELTJES' reply of 19th February 1891:

Je viens justement de recevoir votre objection contre ma démonstration de la proposition de M. TCHEBYCHEFF. Vous vous serez aperçu certainement déjà qu'elle n'est point fondée, après avoir posé

$$\int_a^{a'} \varphi(x)\, dx = (a' - a)\, \varphi(u) \qquad (a < u < a').$$

Je n'ai nullement besoin de la quantité analogue à

$$\int_a^{a'} \psi(x)\,dx = (a'-a)\,\psi(u');$$

mais je considère la valeur de $\psi(x)$ pour $x=u$ (non pour $x=u'$). On a d'abord

$$J=(a'-a)\int_a^{a'} \psi(x)\,[(\varphi(x)-\varphi(u)]\,dx, \tag{1}$$

mais puisque

$$0=\int_a^{a'} [\varphi(x)-\varphi(u)]\,dx, \tag{2}$$

il vient, en multipliant (2) par $(a'-a)\,\psi(u)$ et retranchant de (1),

$$J=(a'-a)\int_a^{a'} [\psi(x)-\psi(u)]\,[\varphi(x)-\varphi(u)]\,dx.$$

En considérant des sommes au lieu d'intégrales, ma démonstration prendrait la forme suivante. Soient

$$a_1,\ a_2,\ \ldots,\ a_n, \qquad b_1,\ b_2,\ \ldots,\ b_n$$

deux séries de n nombres, dont chacune est rangée par ordre de grandeur croissante ou décroissante,

$$J=n\left(\sum_1^n a_k b_k\right)-\left(\sum_1^n a_k\right)\left(\sum_1^n b_k\right).$$

Alors en posant

$$\alpha=\frac{a_1+a_2+\cdots+a_n}{n},$$

c'est-à-dire

$$\sum_1^n (a_k-\alpha)=0,$$

on a aussi

$$J=n\sum_1^n a_k b_k - n\,\alpha\sum_1^n b_k = n\sum_1^n (a_k-\alpha)\,b_k.$$

Supposons maintenant que α tombe entre a_p et a_{p+1}; prenons alors une quantité β entre b_p et b_{p+1} (lorsque par hasard $\alpha=a_p$, on prendrait aussi $\beta=b_p$). Alors à cause de

$$J=n\sum_1^n (a_k-\alpha)\,b_k, \qquad 0=n\sum_1^n (a_k-\alpha)\,\beta,$$

on peut écrire

$$J=n\sum_1^n (a_k-\alpha)\,(b_k-\beta);$$

mais $a_k-\alpha$ et $b_k-\beta$ ont toujours même signe lorsque les séries des a_i et des b_i sont toutes les deux croissantes ou décroissantes, et $a_k-\alpha$ et $b_k-\beta$ ont toujours signes contraires, lorsque l'une des séries est croissante, l'autre décroissente.

Vous voyez que l'introduction de cette quantité β [ou $\psi(u)$] est un point essentiel. L'identité analogue pour la démonstration de M. FRANKLIN, qui se sert d'intégrales doubles, serait

$$\sum_{1}^{n}\sum_{1}^{n}(a_i-a_k)(b_i-b_k)=2n\sum_{1}^{n}(a_i b_i)-2\left(\sum_{1}^{n}a_i\right)\left(\sum_{1}^{n}b_i\right)$$

$$(i,\,k=1,\,2,\,\dots,\,n).$$

The ČEBYŠEV inequality can also be found in another STIELTJES' letter to HERMITE (dated 2nd December 1891):

Vous vous rappelez peut-être le théorème de M. TCHEBYCHEFF sur le signe de l'expression

$$(b-a)\int_a^b f(x)\,\varphi(x)\,\mathrm{d}x-\int_a^b f(x)\,\mathrm{d}x\int_a^b \varphi(x)\,\mathrm{d}x,$$

f et φ étant des fonctions qui varient constamment dans un sens dans l'intervalle $(a,\,b)$.

Voici une généralisation. Soit

$$\mathscr{L}=\int_a^b \psi(x)\,\mathrm{d}x\int_a^b f(x)\,\varphi(x)\,\mathrm{d}x-\int_a^b \varphi(x)\,\mathrm{d}x\int_a^b f(x)\,\psi(x)\,\mathrm{d}x$$

où $f(x)$ est une fonction qui *varie toujours dans un sens*, $\psi(x)$ une fonction qui *reste positive*, tandis que le *rapport* $\dfrac{\varphi(x)}{\psi(x)}$ *varie aussi toujours dans un sens.* Cela étant, on a $\mathscr{L}>0$ lorsque f et $\dfrac{\varphi}{\psi}$ sont toutes les deux croissantes ou toutes les deux décroissantes et $\mathscr{L}<0$ lorsque l'une de ces fonctions est croissante et l'autre décroissante. Pour $\psi(x)=1$, on retombe sur le théorème de M. TCHEBYCHEFF. Voici ma démonstration. On a

$$(1)\qquad \int_a^b f(x)\,\psi(x)\,\mathrm{d}x=f(\xi)\int_a^b \psi(x)\,\mathrm{d}x\qquad (a<\xi<b).$$

Substituons cette valeur dans l'expression $\mathscr{L}$; on peut mettre en facteur l'intégrale $\int_a^b \psi(x)\,\mathrm{d}x$ et il vient

$$(2)\qquad \mathscr{L}=\int_a^b \psi(x)\,\mathrm{d}x\int_a^b [f(x)-f(\xi)]\,\varphi(x)\,\mathrm{d}x.$$

Or, d'après (1),

$$\int_a^b [f(x)-f(\xi)]\,\psi(x)\,\mathrm{d}x=0,$$

le second facteur au second membre de (2) peut donc s'écrire

$$\int_a^b [f(x)-f(\xi)]\,[\varphi(x)-\mathrm{C}\,\psi(x)]\,\mathrm{d}x,$$

C étant une constante quelconque. Prenons $C=\frac{\varphi(\xi)}{\psi(\xi)}$, on aura

$$\mathscr{L}=\int_a^b \psi(x)\,\mathrm{d}x \int_a^b \psi(x)[f(x)-f(\xi)]\left[\frac{\varphi(x)}{\psi(x)}-\frac{\varphi(\xi)}{\psi(\xi)}\right]\mathrm{d}x.$$

Or, si les deux fonctions f et $\frac{\varphi}{\psi}$ varient dans le même sens $f(x)-f(\xi)$ et $\frac{\varphi(x)}{\psi(x)}-\frac{\varphi(\xi)}{\psi(\xi)}$ ont toujours le même signe, et si ces deux fonctions varient en sens inverse $f(x)-f(\xi)$ et $\frac{\varphi(x)}{\psi(x)}-\frac{\varphi(\xi)}{\psi(\xi)}$ auront toujours signe contraire. D'où la proposition énoncée. Je crois qu'on doit attribuer cette proposition à M. JENSEN. En effet, dans le *Bulletin* de Darboux (juin 1888) il y a une Note: *Sur une généralisation d'une formule de M. Tchebycheff*, par M. JENSEN, qui commence ainsi:

«Soient $u_1, u_2, \ldots; v_1, v_2, \ldots; w_1, w_2, \ldots;$ trois suites de grandeurs positives, et telles que l'on ait

$$u_1 \geqq u_2 \geqq u_3 \geqq \cdots \qquad \text{et} \qquad \frac{v_1}{w_1}\geqq\frac{v_2}{w_2}\geqq\frac{v_3}{w_3}\geqq\cdots,$$

on aura toujours

$$\frac{\sum\limits_1^n u_k v_k}{\sum\limits_1^n v_k} > \frac{\sum\limits_1^n u_k w_k}{\sum\limits_1^n w_k}.$$

Suit la démonstration. Ainsi, si $[f(x), u]$, $[\varphi(x), v]$, $[\psi(x), w]$ sont trois fonctions positives telles que $f(x)$ et $\frac{\varphi(x)}{\psi(x)}$ soient constamment décroissantes, on a

$$\frac{\int\limits_a^b f(x)\varphi(x)\,\mathrm{d}x}{\int\limits_a^b \varphi(x)\,\mathrm{d}x} > \frac{\int\limits_a^b f(x)\psi(x)\,\mathrm{d}x}{\int\limits_a^b \psi(x)\,\mathrm{d}x}.$$

Vouz voyez, par ma demonstration, qu'on peut élargir un peu les conditions imposées ici aux fonctions f, φ et ψ.

As can be seen, of all the papers concerning ČEBYŠEV's inequality STIELTJES was only aware of FRANKLIN's and JENSEN's papers.

18. The following result was proved in **[20]** by G. KOWALEWSKI.

Let $x \mapsto f_i(x)$ $(i=1, \ldots, n)$ be real continuous functions of $[a, b]$. Then for n positive numbers $\lambda_1, \ldots, \lambda_n$, such that $\lambda_1+\cdots+\lambda_n=1$ there exist n numbers $t_1, \ldots, t_n \in [a, b]$, such that the following equalities are valid

$$\int_a^b f_i(t)\,\mathrm{d}t=(b-a)\big(\lambda_1 f_i(t_1)+\cdots+\lambda_n f_i(t_n)\big) \qquad (i=1, \ldots, n).$$

This result KOWALEWSKI used in **[21]** to prove ČEBYŠEV's inequality (0.2).

19. SONIN proved inequality (0.1) in [**22**] in the following way. Let η denote the mean

$$\eta=\frac{\int_a^b p(x)\,g(x)\,\mathrm{d}x}{\int_a^b p(x)\,\mathrm{d}x}.$$

Then, the identity

$$\int_a^b p(x)f(x)\,g(x)\,\mathrm{d}x-\frac{\int_a^b p(x)f(x)\,\mathrm{d}x\int_a^b p(x)\,g(x)\,\mathrm{d}x}{\int_a^b p(x)\,\mathrm{d}x}=\int_a^b p(x)\,(f(x)-\alpha)\,(g(x)-\eta)\,\mathrm{d}x,$$

is valid for an arbitrary number α. Since η is the mean value of g, then if g is a nondecreasing function, there exists a number $c\,(a<c<b)$ such that

$$g(x)-\eta\geqq 0 \text{ for } x>c \text{ and } g(x)-\eta\leqq 0 \text{ for } x<c.$$

For $\alpha=f(c+0)$ the product $\big(f(x)-\alpha\big)\big((g(x)-\eta\big)$ has a constant sign. This product is positive if f is a nondecreasing function, or negative if f is a nonincreasing function.

We notice that the quoted proof is connected with that of inequality (0.2) given in STIELTJES' letter to HERMITE of 13 th February 1891 (see **17**).

20. H. BRUNN [**23**] proved inequality (0.2) and also arrived at the same conclusion as H. STIELTJES that the expression

$$(b-a)\int_a^b f(x)\,g(x)\,\mathrm{d}x-\int_a^b f(x)\,\mathrm{d}x\int_a^b g(x)\,\mathrm{d}x$$

is not changed if f and g are replaced by $f+C_1$ and $g+C_1$ respectively, where C_1 is a constant.

BRUNN did not know earlier results in connection with ČEBYŠEV's inequality, and he was informed about them by HURWITZ, as he mentions in [**24**]. Notice that HURWITZ also did not know all the results on ČEBYŠEV's inequality.

21. In [**24**] BRUNN noted that monotony of the functions f and g is sufficient, but not necessary, for the validity of (0.2). He gave, in fact, a different, more general, system of sufficient conditions which ensure the validity of ČEBYŠEV's inequality (0.2). His result is: If for $a\leqq x\leqq b$ we have $\operatorname{sgn}\big(f(x)-f_m\big)=\operatorname{sgn}\big(g(x)-g(x_m)\big)$, where $f_m=\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$, and x_m is determined from $f(x_m)=f_m$, then inequality (0.2) holds.

22. In paper [**25**] M. FUJIWARA gave an apparent generalisation of ČEBYŠEV's inequality. The theorem he proved reads:

Let $f_2(x)\,g_2(x)>0$ in (a, b) and let the functions F and G, defined by

$$(22.1)\qquad F(x, y)=\frac{f_1(x)}{f_2(x)}-\frac{f_1(y)}{f_2(y)} \quad\text{and}\quad G(x, y)=\frac{g_1(x)}{g_2(x)}-\frac{g_1(y)}{g_2(y)}$$

have the same sign for all $x, y\in(a, b)$. Then

$$(22.2)\qquad \int_a^b f_1(x)\,g_1(x)\,dx \int_a^b f_2(x)\,g_2(x)\,dx \geqq \int_a^b f_1(x)\,g_2(x)\,dx \int_a^b f_2(x)\,g_1(x)\,dx.$$

This is only an apparent generalisation of ČEBYŠEV's inequality. Indeed, if we put $f(x)=\dfrac{f_1(x)}{f_2(x)}$, $g(x)=\dfrac{g_1(x)}{g_2(x)}$ and $p(x)=f_2(x)\,g_2(x)$ in (0.1), we get (22.2). Conditions (22.1) can be deduced from ANDRÉIEF's proof.

23. M. FUJIWARA proved in paper [**26**] the following result.

If for $x\in[a, b]$ we have $f_2(x)\,g_2(x)>0$, $\dfrac{g_1(x)}{g_2(x)}\leqq B$ (B an arbitrary positive number), and

$$\frac{f_1(x)}{f_2(x)}\gtrless\frac{\int_a^b f_1(x)\,g_2(x)\,dx}{\int_a^b f_2(x)\,g_2(x)\,dx}$$

then

$$(23.1)\qquad \int_a^b f_1(x)\,g_1(x)\,dx \int_a^b f_2(x)\,g_2(x)\,dx \geqq \int_a^b f_1(x)\,g_2(x)\,dx \int_a^b f_2(x)\,g_1(x)\,dx$$

with equality if and only if $\dfrac{f_1(x)}{f_2(x)}$ and $\dfrac{g_1(x)}{g_2(x)}$ are constants.

Putting $f_1(x)=f(x)$, $g_1(x)=g(x)$, $f_2(x)=g_2(x)=1$, we get the result of BRUNN [**23**] and [**24**].

24. T. HAYAŠHI has proved in [**27**] the discrete inequality (0.4) by applying mathematical induction, assuming that all p_i's $(i=1, \ldots, n)$ have the same sign and that $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are simultaneously increasing or decreasing sequences.

He also quoted that under assumption that all p_i's are of the same sign and that the sequences $(a_{1k}, a_{2k}, \ldots, a_{rk})$ $(k=1, \ldots, r)$ are all either increasing or decreasing, the inequality

$$(24.1)\qquad \left(\sum_{i=1}^n p_i\right)^{r-1} \sum_{i=1}^n p_i\,a_{i1}\,a_{i2}\cdots a_{ir} \geqq \sum_{i=1}^n p_i\,a_{i1}\cdots\sum_{i=1}^n p_i\,a_{ir}$$

holds.

It is easy to see that these conditions are not sufficient for the validity of inequality (24.1) (Counter-example: $p_1=p_2=-1$; $a_{11}=1$, $a_{21}=2$; $a_{12}=1$, $a_{22}=3$; $a_{13}=2$, $a_{23}=3$).

However, (24.1) holds if the assumption on nonnegativity of all appearing sequences is added.

25. J. F. STEFFENSEN **[28]** has proved the theorem:

If F is an increasing function on $[a, b]$ and if F, G and H are integrable functions on $[a, b]$ such that for all $x \in [a, b]$

$$\frac{\int_a^x G(x)\,dx}{\int_a^b G(x)\,dx} \leqq \frac{\int_a^x H(x)\,dx}{\int_a^b H(x)\,dx} \tag{25.1}$$

then

$$\frac{\int_a^b F(x)\,G(x)\,dx}{\int_a^b G(x)\,dx} \geqq \frac{\int_a^b F(x)\,H(x)\,dx}{\int_a^b H(x)\,dx}. \tag{25.2}$$

Putting $F(x)=f(x)$, $H(x)=p(x)>0$, $G(x)=p(x)g(x)$ in (25.1) and (25.2), we conclude that ČEBYŠEV's inequality (0.1) holds if $p(x)>0$, if f is an increasing function on $[a, b]$ and if

$$\frac{\int_a^x p(x)\,g(x)\,dx}{\int_a^x p(x)\,dx} \leqq \frac{\int_a^b p(x)\,g(x)\,dx}{\int_a^b p(x)\,dx}. \tag{25.3}$$

Notice that STEFFENSEN's conditions are weaker than the condition that f and g are both increasing (or both decreasing) functions. Indeed if g is an increasing function, then (25.3) holds, while the converse need not be true.

In the same paper STEFFENSEN proved the corresponding discrete analogue. Namely, he proved that if

$$F(k) \geqq F(k+1), \quad \frac{\sum_{k=n}^{k} G(k)}{\sum_{k=n}^{m} G(k)} \geqq \frac{\sum_{k=n}^{k} H(k)}{\sum_{k=n}^{m} H(k)} \qquad (n \leqq k \leqq m-1),$$

then

$$\frac{\sum_{k=n}^{m} F(k)\,G(k)}{\sum_{k=n}^{m} G(k)} \geqq \frac{\sum_{k=n}^{m} F(k)\,H(k)}{\sum_{k=n}^{m} H(k)}.$$

A special case of this result of STEFFENSEN has recently been rediscovered (see **53**).

26. O. DUNKEL [**29**] has generalised the inequality (0.2) to a system of n functions: If $f_1, \ldots, f_n$ are never negative and experience simultaneously increments, and decrements, then

$$\int_a^b f_1(x)\,dx \cdots \int_a^b f_n(x)\,dx \leqq (b-a)^{n-1} \int_a^b f_1(x) \cdots f_n(x)\,dx. \tag{26.1}$$

However he did not mention the paper [**11**] by ANDRÉIEF.

27. L. BERWALD [**30**] has proved that the inequality (23.1) holds if $\int_a^b f_2 g_2\,dx > 0$ and if there exists a constant B such that for $a \leqq x \leqq b$

$$\Big(f_1(x) - f_2(x) \int_a^b f_1 g_2\,dx : \int_a^b f_2 g_2\,dx\Big)\big(g_1(x) - B g_2(x)\big) \geqq 0.$$

These conditions are weaker than FUJIWARA's conditions.

BERWALD has also proved that the inequality (23.1) is valid if $\int_a^b f_2 g_2\,dx > 0$ and if there exist two constants A and B such that for $a \leqq x \leqq b$

$$\big(f_1(x) - A f_2(x)\big)\big(g_1(x) - B g_2(x)\big) \geqq 0$$

and

$$\left(A - \frac{\int_a^b f_1 g_2\,dx}{\int_a^b f_2 g_2\,dx}\right)\left(B - \frac{\int_a^b f_2 g_1\,dx}{\int_a^b f_2 g_2\,dx}\right) \leqq 0.$$

BERWALD has also extended the first of the above results to the case of n pairs of functions.

28. S. NARUMI [**31**] gave new extensions of the results of FUJIWARA [**25**], [**26**], BERWALD [**30**] and HAYASHI [**27**]:

If $\int_a^b f_2(x) g_2(x)\,dx > 0$,

$$P = \frac{\int_a^b f_1(x) g_2(x)\,dx}{\int_a^b f_2(x) g_2(x)\,dx}, \qquad Q = \frac{\int_a^b f_2(x) g_1(x)\,dx}{\int_a^b f_2(x) g_2(x)\,dx}$$

and if there exist constants A, B, C ($C \geqq 0$) such that for $a \leqq x \leqq b$ we have

$$\big(f_1(x) - P f_2(x)\big)\big(g_1(x) - Q g_2(x)\big) - A\big(g_1(x) - Q g_2(x)\big) f_2(x)$$
$$- B\big(f_1(x) - P f_2(x)\big) g_2(x) - C f_2(x) g_2(x) \geqq 0,$$

then inequality (23.1) holds.

29. Starting from the identity (10.1) for $p(x)=1$ J. CHOKHATE [32] has proved the inequality (0.1).

30. J. CHOKHATE in [33] also used the identity

$$\begin{vmatrix} \int_a^b f_1 g_1 \, d\psi & \cdots & \int_a^b f_1 g_n \, d\psi \\ \vdots & & \\ \int_a^b f_n g_1 \, d\psi & & \int_a^b f_n g_n \, d\psi \end{vmatrix}$$

$$= \frac{1}{n!} \int_a^b \cdots \int_a^b \begin{vmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & & \\ f_n(x_1) & & f_n(x_n) \end{vmatrix} \begin{vmatrix} g_1(x_1) & \cdots & g_1(x_n) \\ \vdots & & \\ g_n(x_1) & & g_n(x_n) \end{vmatrix} d\psi(x_1) \cdots d\psi(x_n)$$

where f_i and g_i are continuous and ψ a monotone nondecreasing function, to derive the ČEBYŠEV inequality for STIELTJES integrals, namely

$$\int_a^b d\psi \int_a^b fg \, d\psi \geqq \int_a^b f \, d\psi \int_a^b g \, d\psi,$$

where f and g are monotone functions in (a, b) in the same sense. This proof derives its origin from the method used in [32].

In the same paper it has also been shown how to derive some other inequalities starting with the aforementioned identity.

31. P. MITRA has proved several inequalities in [34]. The first inequality he obtained, which he called the Fundamental Theorem, reads:

If f and g are two positive continuous functions on $[x_1, x_2]$, then

$$\int_{x_1}^{x_2} \frac{f(x)}{g(x)} \, dx \leqq (x_2 - x_1) \frac{\int_{x_1}^{x_2} f(x) \, dx}{\int_{x_1}^{x_2} g(x) \, dx} \tag{31.1}$$

provided that $x \mapsto \frac{f(x)}{g(x)}$ and $x \mapsto g(x)$ are both increasing or both decreasing functions.

MITRA considers his results to be generalisations of ČEBYŠEV's inequality. He did not notice that (31.1) can be directly deduced from ČEBYŠEV's inequality (0.2). The proof given by MITRA is, in fact, ANDRÉIEF's proof.

In the same paper MITRA has also proved the result:

If $f_1, \ldots, f_n, g_1, \ldots, g_n$ are positive continuous functions on $[x_1, x_2]$, then

$$\int_{x_1}^{x_2} \frac{f_1(x)}{g_1(x)} \, dx \cdots \int_{x_1}^{x_2} \frac{f_n(x)}{g_n(x)} \, dx \leqq (x_2 - x_1)^n \frac{\int_{x_1}^{x_2} f_1(x) \, dx \cdots \int_{x_1}^{x_2} f_n(x) \, dx}{\int_{x_1}^{x_2} g_1(x) \, dx \cdots \int_{x_1}^{x_2} g_n(x) \, dx}$$

provided that some or all functions $x \mapsto \frac{f_1(x)}{g_1(x)}, \ldots, x \mapsto \frac{f_n(x)}{g_n(x)}$ and the corresponding functions $g_1, \ldots, g_n$ are at the same time increasing or decreasing.

This result cannot be directly obtained from ČEBYŠEV's inequality for n functions.

32. According to a review which appeared in Jahrbuch über die Fortschritte der Mathematik **56**, pp. 221—222, in the case of n pairs of functions R.S. VARMA [**35**] has obtained the result analogous to those derived by MITRA [**34**].

33. In paper [**36**] P. MITRA has proved the following result which generalises his previous results:

If $f_1, \ldots, f_n, g_1, \ldots, g_{n-1}$ are positive and continuous functions on $[x_1, x_2]$ such that

$$x \mapsto \frac{f_1(x) \cdots f_{\nu-1}(x)}{g_1(x) \cdots g_{\nu-1}(x)} \quad \text{and} \quad x \mapsto \frac{g_{\nu-1}(x)}{f_\nu(x)} \qquad (\nu = 2, 3, \ldots, n)$$

are both increasing or decreasing functions, then

$$\int_{x_1}^{x_2} \frac{f_1(x) \cdots f_n(x)}{g_1(x) \cdots g_{n-1}(x)} \, \mathrm{d}x \leqq \frac{\int_{x_1}^{x_2} f_1(x)\,\mathrm{d}x \cdots \int_{x_1}^{x_2} f_n(x)\,\mathrm{d}x}{\int_{x_1}^{x_2} g_1(x)\,\mathrm{d}x \cdots \int_{x_1}^{x_2} g_{n-1}(x)\,\mathrm{d}x}.$$

34. In the book [**37**] inequality (0.4) was proved, under assumption that sequences $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are similarly ordered. The identity analogous to (5.1) was used in the proof.

In the same book, p. 168, the following result is quoted:

We say that $f(x, y, \ldots)$ are similarly ordered if

$$\big(f(x_1, y_1, \ldots) - f(x_2, y_2, \ldots)\big)\big(g(x_1, y_1, \ldots) - g(x_2, y_2, \ldots)\big) \geqq 0,$$

oppositely ordered if f and $-g$ are similarly ordered. Then

$$\iint \cdots f \,\mathrm{d}x\,\mathrm{d}y \cdots \iint \cdots g \,\mathrm{d}x\,\mathrm{d}y \leqq \iint \cdots \mathrm{d}x\,\mathrm{d}y \cdots \iint \cdots fg \,\mathrm{d}x\,\mathrm{d}y \cdots$$

if f and g are similarly ordered, while the sign is reversed if f and g are oppositely ordered. The integration is extended over any common part of the regions of f and g.

35. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $z = (z_1, \ldots, z_m)$ and $u = (u_1, \ldots, u_m)$, be given real sequences and let a_{ij} $(i = 1, \ldots, n;\ j = 1, \ldots, m)$ denote real numbers. Then, we have the identity

$$\begin{vmatrix} \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} x_i z_j & \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} x_i u_j \\ \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} y_i z_j & \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} y_i u_j \end{vmatrix}$$

$$= \sum_{1 \leqq i < j \leqq n} \sum_{1 \leqq r < s \leqq m} \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \begin{vmatrix} a_{ir} & a_{is} \\ a_{jr} & a_{js} \end{vmatrix} \begin{vmatrix} z_r & u_r \\ z_s & u_s \end{vmatrix}.$$

As an immediate consequence of the above identity we can formulate the following:

If, for all positive integers i, j, r, s, such that $1 \leqq i < j \leqq n$ and $1 \leqq r < s \leqq m$, we have

$$\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \begin{vmatrix} z_r & u_r \\ z_s & u_s \end{vmatrix} \geqq 0 \text{ and } \begin{vmatrix} a_{ir} & a_{is} \\ a_{jr} & a_{js} \end{vmatrix} \geqq 0, \tag{35.1}$$

then the following inequality holds

$$\frac{\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} x_i z_j}{\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} x_i u_j} \geqq \frac{\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} y_i z_j}{\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} y_i u_j}. \tag{35.2}$$

The last inequality is due to G. SEITZ [38]. The text of the paper [38] is rather confused and contains a number of mistakes in indices.

For $y_i = u_i = 1$, $a_{ik} = 0\,(i \neq k)$ and $a_{ii} = p_i > 0$ inequality (35.2) reduces to ČEBYŠEV's inequality (0.4).

If $m = n$, $x_i = z_i$, $y_i = u_i$ $(i = 1, \ldots, n)$ and $a_{ik} = \delta_{ik}$ (δ_{ik} is the KRONECKER symbol), then (35.2) becomes CAUCHY's inequality.

In the same paper SEITZ has also given the integral analogue to (35.2).

36. N. CIORANESCU [39] has obtained the inequality (0.1) in exactly the same way as ANDRÉIEF [11] without even mentioning that this is the ČEBYŠEV inequality or quoting any literature.

37. In paper [40] J. A. SHOHAT and A. V. BUSHKOVITCH have applied inequality (0.1) to remainders which are obtained in developing certain functions in TAYLOR's series, and have arrived at a number of inequalities for various elementary functions.

38. R. LAGUARDIA in paper [41] again arrived at ČEBYŠEV's inequality for STIELTJES' integrals, which had already been obtained by J. SHOHAT [33]. R. LAGUARDIA mentions only book [37] and FRANKLIN's paper [13].

39. D. N. LABUTIN [42] has examined under which conditions the inequality

$$\int_a^x \frac{f(x)}{g(x)}\,\mathrm{d}x < (x-a) \frac{\int_a^x f(x)\,\mathrm{d}x}{\int_a^x g(x)\,\mathrm{d}x}, \tag{39.1}$$

holds, where f and g are continuous and positive functions on the considered segment.

Inequality (39.1) is, in fact, one of the forms in which ČEBYŠEV's inequality (0.2) can be represented, though LABUTIN makes no mention of that.

Labutin has proved that (39.1) holds if one of the following conditions (denoted by 1°— 6°) is fulfilled for all values of x from the considered interval:

1° $\int_a^x g(x)\,dx<(x-a)g(x), \quad g(x)\int_a^x f(x)\,dx<f(x)\int_a^x g(x)\,dx;$

2° $\int_a^x g(x)\,dx>(x-a)g(x), \quad g(x)\int_a^x f(x)\,dx>f(x)\int_a^x g(x)\,dx;$

3° In some of the intervals, $[a, a_1]$ $[a, a_2]$, ..., $[a, x]$ contained in the given interval, the conditions 1° are fulfilled and in the rest of them, the conditions 2°;

4° Functions $x\mapsto g(x)$ and $x\mapsto \frac{f(x)}{g(x)}$ are both increasing or both decreasing in the considered interval;

5° Functions $x\mapsto g(x)$ and $x\mapsto \frac{f(x)}{g(x)}$ are both increasing (decreasing) in $[a, c]$ and decreasing (increasing) in other parts of the interval. In addition, those two functions are even functions;

6° The derivatives of f and g exist and $f'(x)>g'(x)>0$, $g(x)>f(x)$, or $f'(x)<g'(x)<0$, $g(x)>f(x)$.

In the same paper Labutin has given a set of sufficient conditions for the validity of the discrete analogue of (39.1).

40. D. N. Labutin has published another paper [**43**] on a similar topic, and has given sufficent conditions for the validity of

$$(b-a)\int_a^b \frac{dt}{u(t)}>\int_a^b \frac{dt}{v(t)}\int_a^b \frac{v(t)}{u(t)}\,dt. \tag{40.1}$$

It is not difficult to see that from (39.1) for $f(x)=\frac{1}{u(x)}$, $g(x)=\frac{1}{v(x)}$ follows (40.1).

41. M. Biernacki [**44**] has proved that inequality (0.1) holds also in the case when the functions

$$x\mapsto f(x) \quad \text{and} \quad x\mapsto \frac{\int_a^x p(x)\,g(x)\,dx}{\int_a^x p(x)\,dx}$$

are both increasing or both decreasing.

These conditions are similar to the ones given by Steffensen [**28**], but are, in fact, less general. Indeed, Steffensen requires only that

$$\frac{\int_a^x p(x)\,g(x)\,dx}{\int_a^x p(x)\,dx}\leq \frac{\int_a^b p(x)\,g(x)\,dx}{\int_a^b p(x)\,dx}.$$

42. N. A. SAPOGOV [45] has extended ČEBYŠEV's inequality by considering abstract integrals on a set E, and at the same time has extended the conditions for the validity of ČEBYŠEV's inequality. His result reads:

Let the real functions $u, v, w, x \mapsto u(x)v(x)$ and $x \mapsto u(x)w(x)$ be integrable on E, and let e denote the set

$$\left\{x \,\middle|\, v(x) \int_E w(x)\,d\mu - w(x) \int_E v(x)\,d\mu > 0\right\}.$$

If for arbitrary $x' \in e$, $x'' \in E$ we have $u(x') \geqq u(x'')$, then

$$\int_E w(x)\,d\mu \int_E u(x)v(x)\,d\mu \geqq \int_E v(x)\,d\mu \int_E u(x)w(x)\,d\mu.$$

In the same paper SAPOGOV gave a simpler proof of inequality (0.2). This proof is in a way similar to the proofs of STIELTJES [19] and SONIN [22].

43. In paper [46] M. BIERNACKI has proved that inequality (0.1) holds if p, f, g are integrable functions in (a, b), such that $p(x) > 0$ $\big(x \in (a, b)\big)$, and if the functions f_1 and g_1, given by

$$f_1(x) = \frac{\int_a^x p(x) f(x)\,dx}{\int_a^x p(x)\,dx}, \quad g_1(x) = \frac{\int_a^x p(x) g(x)\,dx}{\int_a^x p(x)\,dx}$$

reach extreme values in (a, b) in a finite number of common points, and are also both increasing or both decreasing in (a, b). If one of the functions f_1, g_1 is increasing and the other decreasing, then the sign of inequality is reversed in (0.1).

44. In 1952 KY FAN [47] proposed as a problem the following result:

Let $(x, y) \mapsto K(x, y)$ be a nonnegative Lebesgue integrable function over the square $\{(x, y) \mid a \leqq x \leqq b \text{ and } a \leqq y \leqq b\}$. Suppose that B is a positive constant such that $\int_a^b K(x, y)\,dy \leqq B$ for almost all x from $[a, b]$, and also $\int_a^b K(x, y)\,dx \leqq B$ for almost all y from $[a, b]$. If two finite-valued functions f and g are both nonnegative and nonincreasing on $[a, b]$, then

$$\int_a^b \int_a^b K(x, y) f(x) g(y)\,dx\,dy \leqq B \int_a^b f(x) g(x)\,dx. \tag{44.1}$$

If $K(x, y) = \text{const}$, (44.1) reduces to (0.2).

The solution to this problem was given by the author himself. The fact that no other solution arrived (though for some problems published in the same journal dozens of mathematicians send solutions) shows that certain problems have solutions which are „more difficult" than certain papers.

45. H. SCHWERDTFEGER has proved in [**48**] the following result (see also [**49**]):

If f is a nonnegative and nonincreasing function over $0 \leqq x \leqq 1$ then

$$\frac{\int_0^1 x f(x)^2 \, \mathrm{d}x}{\int_0^1 x f(x) \, \mathrm{d}x} \leqq \frac{\int_0^1 f(x)^2 \, \mathrm{d}x}{\int_0^1 f(x) \, \mathrm{d}x}. \tag{45.1}$$

This inequality represents, in fact, a special case of an integral analog of inequality (1.1).

The inequality (45.1) may be derived from (0.1) $\big(p(x) = F(x)^2, f(x) = x, g(x) = 1/F(x)\big)$.

46. Supposing that f, g and p are positive functions on $[a, b]$ P. LOVERA [**50**] obtained inequality (0.1). Clearly he also did not know about earlier results concerning that inequality, since he only mentioned the book [**51**], for which he says that it contains a proof of (0.1) under more restrictive suppositions. P. LOVERA refers to (0.1) as JENSEN-STEFFENSEN inequality. Besides, LOVERA's proof is identical with the proof of ANDRÉIEF [**11**] for the case $n = 2$.

In the same paper the following inequality has also been proved: If f and g are positive functions on $[a, b]$ and f is monotone increasing on $[a, b]$, then for $a < c < b$ we have

$$\frac{\int_a^c f(x) g(x) \, \mathrm{d}x}{\int_a^c g(x) \, \mathrm{d}x} < \frac{\int_a^b f(x) g(x) \, \mathrm{d}x}{\int_a^b g(x) \, \mathrm{d}x} < \frac{\int_c^b f(x) g(x) \, \mathrm{d}x}{\int_c^b g(x) \, \mathrm{d}x}.$$

47. The results mentioned up to now were concerned with 1° giving new proofs of ČEBYŠEV's inequalities and 2° determining weaker conditions which ensure the validity of those inequalities.

Another possibility would be to examine whether stronger conditions will produce sharper inequalities. One of the results of that kind was obtained by B. J. ANDERSSON [**52**] who proved that if $f_1, \ldots, f_n$ are convex nonnegative functions defined on [0,1] such that $f_k(0) = 0$ $(k = 1, \ldots, n)$, then

$$\int_0^1 f_1(x) \cdots f_n(x) \, \mathrm{d}x \geqq \frac{2^n}{n+1} \Big(\int_0^1 f_1(x) \, \mathrm{d}x\Big) \cdots \Big(\int_0^1 f_n(x) \, \mathrm{d}x\Big). \tag{47.1}$$

Since $2^n \geqq n+1$, inequality (46.1) is sharper than inequality (0.3), i.e. than (26.1) for $a = 0$ and $b = 1$.

48. The following result of T. POPOVICIU [**51**] is closely connected to ČEBYŠEV's inequality (0.4).

Let $a=(a_1, \ldots, a_n)$ and $b=(b_1, \ldots, b_n)$ be nondecreasing sequences of real numbers and let x_{ij} $(i, j=1, \ldots, n)$ be real numbers. Then necessary and sufficient conditions for the numbers x_{ij}, so that the inequality

$$(48.1) \qquad F(a, b)=\sum_{i,j=1}^{n} x_{ij} a_i b_j \geqq 0$$

holds: 1° for all nondecreasing sequences a and b, or 2° for all nonnegative nondecreasing sequences a and b, are contained in the following two theorems.

1° *With the condition* 1°, $F(a, b) \geq 0$ *if and only if*

$$\sum_{i=r}^{n} \sum_{j=s}^{n} x_{ij} \geqq 0 \qquad (r=1, \ldots, n;\ s=2, \ldots, n),$$

$$\sum_{i=r}^{n} \sum_{j=1}^{n} x_{ij}=0 \qquad (r=1, \ldots, n).$$

2° *With the condition* 2°, $F(a, b) \geqq 0$ *if and only if*

$$\sum_{=r}^{n} \sum_{j=s}^{n} x_{ij} \geqq 0 \qquad (r=1, \ldots, n;\ s=1, \ldots, n).$$

Remark. Theorems 1° and 2° refer only to the cases of nondecreasing sequences (Theorem 1°) and nonnegative nondecreasing sequences (Theorem 2°). Similar results are obtained in other cases.

Inequality (48.1) is a generalisation of a number of known inequalities. For example, taking

$$x_{ij}=n-1 \quad (i=j), \qquad x_{ij}=-1 \quad (i \neq j),$$

we get the inequality (0.4).

49. The condition that the sequences $a=(a_1, \ldots, a_n)$ and $b=(b_1, \ldots, b_n)$ are similarly ordered is only a sufficient condition for the validity of ČEBYŠEV's inequality (0.5). Some other sufficient conditions for the validity of (0.5) were given by LABUTIN [**42**]. A set of necessary and sufficient conditions was given in [**54**] by D. W. SASSER and M. L. SLATER. Their result reads:

Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be two sequences of real numbers. Let a and b be n-dimensional column vectors whose components are $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ respectively. Furthermore, let be an n-dimensional column vector with all entries equal to 1. *Then, a necessary and sufficient condition for Čebyšev's inequality to hold is $b=Aa+ce$ or $a=Ab+ce$, where c is a real positive semi-definite matrix, such that the sum of the elements of any column or row is* 0. *Equality occurs in* (0.5) *if and only if* $(A+A')a=0$ *or* $(A+A')b=0$.

50. BERLJAND, NAZAROV and SVIDSKIĬ [**55**] have formulated and sketched the proof of the following generalisation of ČEBYŠEV's inequality:

Let P, Q be real-valued continuously differentiable functions on a region $\Omega \subset \mathbf{R}^n$, and let $s: \mathbf{R}^n \rightarrow \mathbf{R}^1$ be continuous and nonnegative on Ω. Then the inequality

$$(50.1) \qquad \int_{\omega} P(\bar{x}) G(\bar{x}) s(\bar{x}) \, d\bar{x} \geqq \int_{\omega} P(\bar{x}) s(\bar{x}) \, d\bar{x} \int_{\omega} Q(\bar{x}) s(\bar{x}) \, d\bar{x}$$

holds for every subregion $\omega \subset \Omega$ if, and only if, P and Q are functionally dependent on Ω, $P = f(Q)$ or $Q = f(P)$, where f is nondecreasing. If f is nonincreasing, the inequality sign in (50.1) is reversed.

In Zentralblatt für Mathematik, vol. 173 (1969), P. R. BEESACK has given the following comment on the above result:

„The proof is based on the fact that (50.1) is equivalent to

$$\Delta = \frac{1}{2} \int_\omega \int_\omega s(\bar{x})\, s(y)\, [P(\bar{x}) - P(\bar{y})]\, [Q(\bar{x}) - Q(\bar{y})]\, d\bar{x}\, d\bar{y} \geqq 0. \tag{50.2}$$

The sufficiency of the conditions on P, Q are therefore immediate even for P, Q continuous on Ω. In the sketch of the proof of necessity, the idea is to show first that if P, Q are not functionally dependent on Ω, then the integrand in (50.2) is negative. Although details are not given, this can be shown to be the case for a certain region $\Omega \times \Omega$. But the proof appears to be incomplete since the integrand must be shown to be negative on a set of the form $\omega \times \omega$ in order to obtain a contradiction. This cannot be done for ω a cell in R^n. A counterexample for the case $n = 2$ is given by taking $\Omega = (0,1) \times (0,1)$ for example, and $P(x_1, x_2) = x_1$, $Q(x_1, x_2) = x_2$, $s(x_1, x_2) = 1$, so P, Q are functionally independent; nevertheless for any cell $(a, b) \times (c, d) = \omega \subset \Omega$, equality holds in (50.1), so $\Delta = 0$ for such ω. Actually equality holds in (50.1) in this case for any square $\omega \subset \Omega$, whether its sides are parallel to the axes or not. This example is not, however, a counterexample for the necessity part of the theorem (but only for its apparent proof) since inequality (50.1) is violated for certain sets $\omega \subset \Omega$. The inequality (50.1), with the trivial change that $s \equiv 1$, is not new, but appears as Theorem 236 in G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA [**37**]. Here, the result is only stated as a sufficient condition for (50.1), namely that (50.1) holds if P, Q are similarly ordered on Ω. In case P, $Q \subset C^1(\Omega)$ the authors have actually proved that P, Q are similarly ordered on Ω if, and only if, P and Q are functionally dependent on Ω, with P a nondecreasing function of Q or vice-versa.

51. H. M. MCLAUGHLIN and F. T. METCALF [**56**] have obtained an interesting result which generalises ČEBYŠEV's inequality. Their result reads:

Let I and J denote nonempty disjoint finite sets of distinct positive integers. Suppose that (a_k) and (b_k), with $k \in I \cup J$, are sequences of nonnegative real numbers, (p_k), with $k \in I \cup J$, is a sequence of positive numbers, and $r > 0$. Define M_r and T_r by

$$M_r(a; p, I) = \left(\frac{\sum_{k \in I} p_k a_k{}^r}{\sum_{k \in I} p_k} \right)^{\frac{1}{r}}$$

and

$$T_r(a, b; I) = \Big(\sum_{k \in I} p_k \Big) \big(M_r(ab; p, I)^r - M_r(a; p, I)^r M_r(b; p, I)^r \big).$$

If the pairs

$$(M_r(a; p, I),\ M_r(a; p, J)) \quad \textit{and} \quad (M_r(b; p, I),\ M_r(b; p, J)) \tag{51.1}$$

are similarly ordered, then

$$T_r(a, b; I \cup J) \geqq T_r(a, b; I) + T_r(a, b; J). \tag{51.2}$$

If the pairs (51.1) *are oppositely ordered, then the sense of* (51.2) *reverses. In both cases equality holds if and only if either*

$$M_r(a; p, I) = M_r(a; p, J) \quad \text{or} \quad M_r(b; p, I) = M_r(b; p, J).$$

If we put $I = \{1, \ldots, n\} = I_n$ and $J = \{n+1\}$, from (51.2) follows

$$T_r(a, b; I_{n+1}) \geqq T_r(a, b; I_n)$$

which implies

$$T_r(a, b; I_{n+1}) \geqq T_r(a, b; I_n) \geqq \cdots \geqq T_r(a, b; I_1) = 0,$$

i.e. inequality (0.4).

52. A. OSTROWSKI ([**57**]) considered the expression

$$T(f, g) = M(f, g) - M(f)\, M(g),$$

where $M(f)$ is a „general mean", defined in the following way:

Let Φ be a space of real-valued functions $f(P)$ for all P running through the argument space S, and assume that Φ is an algebra over the field R of real numbers. Further, assume that all real constans belong to Φ.

Consider now a real-valued functional $M(f)$ defined on Φ such that:

1° $(\forall \alpha, \beta \in R, \forall f, g \in \Phi)$ $M(\alpha f + \beta g) = \alpha M(f) + \beta M(g)$,

2° $(\forall f \in \Phi)$ $M(f) \geqq 0$ $(f \geqq 0)$,

3° $(\forall \alpha \in R)$ $M(\alpha) = \alpha$.

We shall say that the functions $f, g \in \Phi$ are synchronic if for any pair of points $P, Q \in S$ we have

$$f(P) \geqq f(Q) \text{ if and only if } g(P) \geqq g(Q).$$

Then, if $f(P)$, $g(P)$ are synchronic, we have

$$T(f, g) \geqq 0. \tag{52.1}$$

For $M(f)$ we may take, for instance,

$$M(f) = \frac{\int_a^b p(x) f(x)\, dx}{\int_a^b p(x)\, dx}.$$

Then inequality (52.1) implies inequality (0.1).

53. J. HERSCH and M. MONKEWITZ [**58**] in order to generalise an isoperimetric inequality due to SZEGÖ and WEINBERGER, have previously proved the following result:

If g is a nondecreasing on [0, 1] and if f satisfies

$$\frac{1}{x^2}\int_0^x tf(t)\,\mathrm{d}t \leqq \int_0^1 tf(t)\,\mathrm{d}t \qquad (0<x<1),$$

then

$$\int_0^1 tf(t)\,g(t)\,\mathrm{d}x \geqq 2\int_0^1 tf(t)\,\mathrm{d}t \int_0^1 tg(t)\,\mathrm{d}t.$$

This result is a special case of STEFFENSEN's result (see [**28**]) for $p(x)=x$.

54. In 1971 F. V. ATKINSON [**59**] has proved that the inequality (0.2) holds if $f(x)$, $g(x)\in C''[a, b]$, $f''(x)>0$, $g''(x)>0$ and

$$\int_a^b \Big(x-\frac{1}{2}(a+b)\Big)g(x)\,\mathrm{d}x=0.$$

55. A. LUPAS [**60**] has proved the following theorem: If f, g are convex functions on the interval $[a, b]$, then

$$\int_a^b f(x)\,g(x)\,\mathrm{d}x-\frac{1}{b-a}\Big(\int_a^b f(x)\,\mathrm{d}x\Big)\Big(\int_a^b g(x)\,\mathrm{d}x\Big)$$

$$\geqq\frac{12}{(b-a)^2}\Big(\int_a^b\Big(x-\frac{a+b}{2}\Big)f(x)\,\mathrm{d}x\Big)\Big(\int_a^b\Big(x-\frac{a+b}{2}\Big)g(x)\,\mathrm{d}x\Big),$$

with equality when at least one of the function f and g is a linear function on $[a, b]$.

A corollary of this theorem is: Let f, g be convex functions on $[a, b]$ and assume that $g\Big(\frac{a+b}{2}-x\Big)=g\Big(\frac{a+b}{2}+x\Big)\Big(x\in\Big[-\frac{b-a}{2}, \frac{b-a}{2}\Big]\Big)$. Then inequality (0.2) holds.

The result of F. V. ATKINSON (see **54**) also follows from the above theorem.

56. In [**61**] the following generalisation of ANDERSSON's result (see **47**) is proved: If $f_1, \ldots, f_n$ are integrable and convex functions on $[a, b]$ such that $f_k(x)\geqq 0$, $p(x)\geqq 0$ for $x\in[a, b]$, and $f_k(a)=0$ $(k=1, \ldots, n)$, then inequality

$$\text{(56.1)}\qquad \Big(\int_a^b p(x)\,\mathrm{d}x\Big)^{n-1}\Big(\int_a^b p(x)f_1(x)\cdots f_n(x)\,\mathrm{d}x\Big)$$

$$\geqq M\Big(\int_a^b p(x)f_1(x)\,\mathrm{d}x\Big)\cdots\Big(\int_a^b p(x)f_n(x)\,\mathrm{d}x\Big)$$

holds, where

$$M=\frac{\left(\int_a^b p(x)(x-a)^n\,dx\right)\left(\int_a^b p(x)\,dx\right)^{n-1}}{\left(\int_a^b p(x)(x-a)\,dx\right)^n}.$$

Equality holds if and only if $f_k(x)=c_k(x-a)$ $(k=1, \ldots, n)$.

Inequality (56.1) for $n=2$ is sharper than (0.1) but the convexity condition is added.

57. In paper **[62]** P. M. VASIĆ and R. Ž. ĐORĐEVIĆ have proved the following two theorems:

Theorem 1. *If $A=(A_1, \ldots, A_n)$ and $B=(B_1, \ldots, B_n)$ are two real sequences such that*

$$A_1 \geqq \cdots \geqq A_n \text{ and } B_1 \geqq \cdots \geqq B_n,$$

or

$$A_1 \leqq \cdots \leqq A_n \text{ and } B_1 \leqq \cdots \leqq B_n,$$

then the following inequality holds

$$T_n(A, B; P) \geqq T_{n-1}(A, B; P) \tag{57.1}$$

where

$$T_n(A, B; P)=\sum_{i=1}^n P_i \sum_{i=1}^n P_i A_i B_i-\sum_{i=1}^n P_i A_i \sum_{i=1}^n P_i B_i$$

and $P=(P_1, \ldots, P_n)$ is a positive sequence.

Equality in (57.1) *holds if and only if $A_i=A_n$ $(i \in I \subset \{1, \ldots, n\})$, $B_j=B_n$ $(j \in \{1, \ldots, n\} \setminus I)$ while A_i $(i \in \{1, \ldots, n\} \setminus I)$ and B_j $(j \in I)$ are arbitrary.*

Theorem 2. *If $a=(a_1, \ldots, a_n)$ and $b=(b_1, \ldots, b_n)$ are two real sequences such that*

$$0=a_1 \leqq \cdots \leqq a_n, \qquad 0=b_1 \leqq \cdots \leqq b_n,$$

$$\left.\begin{array}{l} a_{i-1}-2a_i+a_{i+1} \geqq 0 \\ b_{i-1}-2b_i+b_{i+1} \geqq 0 \end{array}\right. \quad (i=2, \ldots, n-1),$$

then the following inequality is valid

$$C_n(a, b; p) \geqq C_{n-1}(a, b; p), \tag{57.2}$$

where

$$C_n(a, b; p)=\left\{\sum_{i=1}^n p_i(i-1)\right\}^2 \sum_{i=1}^n p_i a_i b_i-\sum_{i=1}^n p_i(i-1)^2 \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i$$

and $p=(p_1, \ldots, p_n)$ is a positive sequence.

Equality in (57.2) *holds if and only if $a_1=\cdots=a_n$, $b_1=\cdots=b_n$.*

Since $a_1=b_1=0$, $C_2(a, b; p)=0$, from (57.2) we get,

$$C_n(a, b; p) \geqq C_{n-1}(a, b; p) \geqq \cdots \geqq C_2(a, b; p)=0,$$

so that we can write

$$\left\{\sum_{i=1}^{n} p_i(i-1)\right\}^2 \sum_{i=1}^{n} p_i a_i b_i \geqq \sum_{i=1}^{n} p_i (i-1)^2 \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i. \tag{57.3}$$

Since, using the Čebyšev inequality, we have

$$\frac{\sum_{i=1}^{n} p_i (i-1)^2}{\left\{\sum_{i=0}^{n} p_i (i-1)\right\}^2} \geqq \frac{1}{\sum_{i=1}^{n} p_i},$$

it may be concluded that inequality (57.3) is sharper than the Čebyšev inequality, provided that the conditions of convexity and nonnegativity for the sequences a and b are added.

58. The following result is a natural extension of inequality (0.5): If $a_{i_1\ldots i_r}$ and $b_{i_1\ldots i_r}$ $(k=1, \ldots, r,\ i_k=1, \ldots, m_k)$ are real functions of indices $i_1, \ldots, i_r$, then inequality

$$\prod_k m_k \sum_i a_i b_i \geqq \sum_i a_i \sum_i b_i \tag{58.1}$$

where the summation is made over all combinations of indices $i=(i_1, \ldots, i_r)$ is valid if

$$(a_{i_1\ldots i_r} - a_{j_1\ldots j_r})(b_{i_1\ldots i_r} - b_{j_1\ldots j_r}) \geqq 0$$

for every $(i_1, \ldots, i_r)$ and $(j_1, \ldots, j_r)$.

L. Vietoris [**63**] has shown that inequality (58.1) is valid even under other conditions. Namely, he has shown that (58.1) holds if for every two combinations $(i_1, \ldots, i_r), (j_1, \ldots, j_r)$ for which $i_k \leqq j_k$ $(k=1, \ldots, r)$, $a_{i_1\ldots i_r} \leqq a_{j_1\ldots j_r}$ is also satisfied.

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