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444. ON ENUMERATION OF CERTAIN TYPES OF SEQUENCES*

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1. Introduction

In this paper we shall consider a problem, known in literature (see [**1**]), but with a different approach, which will lead to some special results. Primarily, we shall define the problem in the same way as in the original paper.

Let n be a fixed positive integer and let $f_j(n)$ denote the number of sequences of nonnegative integers

$$(a_1, \ldots, a_n), \tag{1.1}$$

such that

$$|a_i - a_{i+1}| = 1 \qquad (i = 1, \ldots, n-1) \tag{1.2}$$

and

$$a_1 = j. \tag{1.3}$$

Also let $f_{j,k}(n)$ denote the number of sequences (1.1), that satisfy (1.2) and

$$a_1 = j, \qquad a_n = k. \tag{1.4}$$

Next, let $g_j(n)$ denote the number of sequences (1.1) satisfying (1.3) and

$$|a_i - a_{i+1}| \leqq 1 \qquad (i = 1, \ldots, n-1); \tag{1.5}$$

let $g_{j,k}(n)$ denote the number of sequences (1.1), that satisfy (1.4) and (1.5).

Now, we shall quote the main results from [**1**].

We have

$$f_k(n+1) = \sum_{2j \leqq k} (-1)^j \binom{k-j}{j} \binom{n+k-2j}{[(n+k-2j)/2]} \tag{1.6}$$

or

$$\begin{aligned} f_k(n+1) &= 2^n, \qquad 0 \leqq n \leqq k, \\ f_{n-k}(n+1) &= 2^n - P_k(n), \quad 0 \leqq k \leqq n, \end{aligned} \tag{1.7}$$

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* Presented June 10, 1973 by L. Carlitz.

where $P_k(n)$ is defined as

$$P_{2k}(n)=2\sum_{j=0}^{k-1}\binom{n}{j},\quad P_{2k+1}(n)=\binom{n}{k}+2\sum_{j=0}^{k-1}\binom{n}{j}. \tag{1.8}$$

The corresponding result for $f_{j,k}(n)$ is given by

$$f_{j,k}(n+1)=\begin{cases}\sum_{2s\leqq j}(-1)^s\binom{j-s}{s}\left\{\binom{n+j-2s}{(n+j-k-2s)/2}-\binom{n+j-2s}{(n+j-k-2s-2)/2}\right\}, & n\equiv j+k \pmod 2\\ 0, \quad n\equiv j+k+1 \pmod 2. & \end{cases} \tag{1.9}$$

Next, the coefficients $c(m, k)$ have been introduced by means of the generating function

$$(1+x+x^2)^m=\sum_{k=0}^{\infty} c(m, k)\, x^k, \tag{1.10}$$

where m is an arbitrary integer. The result for $g_j(n)$ is:

$$g_k(n+1)=\sum_{j=0}^{k} c(-j-1,\ k-j)\{c(n+j,\ n+j)+c(n+j,\ n+j+1)\}; \tag{1.11}$$

also, corresponding to (1.7), formulas

$$\begin{aligned} g_k(n+1) &= 3^n, \qquad 0\leqq k\leqq n,\\ g_{n-k}(n+1) &= 3^n-Q_k(n), \qquad 0\leqq k\leqq n, \end{aligned} \tag{1.12}$$

have been given, where $Q_0(n)=0$, $Q_1(n)=1$ and

$$Q_{k+1}(n+1)=c(n, k)+2\sum_{j=0}^{k-1} c(n, j). \tag{1.13}$$

And at last we arrive at

$$g_{j,k}(n+1)=\sum_{s=0}^{j} c(-s-1,\ j-s)\{c(n+s, n+s-k)-c(n+s, n+s-k-2)\}. \tag{1.14}$$

In this paper we shall derive the expressions for all the above functions, and we shall also give generating functions for them. As a consequence of the way of solving the above problem, we shall obtain some combinatorial identities.

2. Formulation of the problem in the graph theory

Consider a labelled chain G_1 of the length m (this is a connected graph with m vertices, two of which being of degree 1 and others of degree 2), where $m=n+j+p\ (p\geqq 0)$. The vertices of G_1 are labelled in natural manner starting from 0 in one end-point and finishing at $m-1$ in the other one.

It is easy to see that the number of walks of length $n-1$ in G_1 starting in the vertex j is equal to $f_j(n)$. $f_{j,k}(n)$ is, in accordance with the stated, the number of such walks but between vertices j and k.

For $g_j(n)$ and $g_{j,k}(n)$ the foregoing holds, too, but, instead of G_1, the graph G_2, obtained from G_1 by adding a loop to each of vertices of G_1, is considered.

It is well known that the element at the place (j, k) of the n-th power of the adjacency matrix of a graph G is equal to the number of walks of length n in G starting from the vertex j and terminating in the vertex k.

If we correspond the vertex labelled by $i-1$ $(i=1,2,\ldots,m)$, to the i-th row (or column) of the adjacency matrix, we get

$$(2.1)\qquad \begin{aligned} f_{j,k}(n+1)&=a_{j+1,\,k+1}^{(n)}, & f_j(n+1)&=\sum_k a_{j+1,\,k+1}^{(n)},\\ g_{j,k}(n+1)&=\overset{*}{a}{}_{j+1,\,k+1}^{(n)}, & g_j(n+1)&=\sum_k \overset{*}{a}{}_{j+1,\,k+1}^{(n)}\end{aligned}$$

where $A^n=\left\|a_{p,q}^{(n)}\right\|$ and $\overset{*}{A}{}^n=\left\|\overset{*}{a}{}_{p,q}^{(n)}\right\|$, A and $\overset{*}{A}$ being adjacency matrices of G_1 and G_2 respectively.

The adjacency matrix A of an undirected graph (having m vertices) is symmetric and so there exists the system of eigenvectors $u_1,\ldots,u_m$ belonging to eigenvalues $\lambda_1\ldots,\lambda_m$ of A, such that each eigenvector is orthogonal to each other. If these vectors are normalized, so that their moduli are equal to 1, the matrix $U=\|u_1\cdots u_m\|$, whose columns are the mentioned eigenvectors, satisfies the relation $A=U\Lambda U^{-1}$, where Λ is a diagonal matrix with diagonal elements $\lambda_1,\ldots,\lambda_m$. Since the matrix U is orthogonal (i.e $U^{-1}=U^T$), we have $A^k=U\Lambda^k U^T$.

If we take $U=\|u_{ij}\|$ (i.e. eigenvector u_j has the components u_{ij}), we get

$$(2.2)\qquad a_{ij}^{(k)}=\sum_{l=1}^{m} u_{il}\,u_{jl}\,\lambda_l^k.$$

It is known that eigenvalues of a chain of length m are given by

$$(2.3)\qquad \lambda_i=2\cos\frac{i\pi}{m+1}\qquad (i=1,\ldots,m),$$

while

$$(2.4)\qquad u_{ij}=\sqrt{\frac{2}{m+1}}\sin\frac{ij\pi}{m+1}\qquad (i,j=1,\ldots,m)$$

represents the corresponding eigenvectors (see, for example, [2]).

3. Determination of $f_{j,k}(n)$, $g_{j,k}(n)$, $f_j(n)$, $g_j(n)$

3.1. We shall now deduce a formula for $f_{j,k}(n)$. From (2.1), using (2.2), (2.3) and (2.4), we arrive at

$$(3.1)\qquad f_{k,j}(n+1)=\frac{2}{n+j+p+2}\sum_{l=1}^{n+j+p+1}\sin\frac{(j+1)l\pi}{n+j+p+2}\sin\frac{(k+1)l\pi}{n+j+p+2}\times\left(2\cos\frac{l\pi}{n+j+p+2}\right)^n.$$

Since p is an arbitrary nonnegative integer, it is of interest to take $p \to +\infty$. Then from (3.1) it follows:

$$f_{j,k}(n+1)=\frac{1}{\pi}\int_{-\pi}^{\pi}\sin(j+1)x\sin(k+1)x\,(2\cos x)^n\,dx. \tag{3.2}$$

It is easy to show by the calculus of residues that integral (3.2) is equal to

$$f_{k,j}(n+1)=\begin{cases}\binom{n}{(n-j+k)/2}-\binom{n}{(n-j-k-2)/2}, & n\equiv j+k \pmod 2\\ 0, & n\equiv j+k+1 \pmod 2).\end{cases} \tag{3.3}$$

3.2. Now, we shall repeat the above procedure for the function $g_{j,k}(n)$. We have obviously,

$$g_{j,k}(n+1)=\frac{2}{n+j+p+2}\sum_{l=1}^{n+j+p+1}\sin\frac{(j+1)l\pi}{n+j+p+2}\sin\frac{(k+1)l\pi}{n+j+p+2} \tag{3.4}$$

$$\times\left(2\cos\frac{l\pi}{n+j+p+2}+1\right)^n=\frac{1}{\pi}\int_{-\pi}^{\pi}\sin(j+1)x\sin(k+1)x\,(2\cos x+1)^n\,dx$$

and

$$g_{j,k}(n+1)=c(n,n+j-k)-c(n,n-j-k-2). \tag{3.5}$$

3.3. Formulas for $f_j(n)$ and $g_j(n)$ are direct consequences of the former ones. Using formulas (2.1) and (3.3) (or (3.5)) it is easy to obtain the results (1.7) (or (1.12)) from [1].

3.4. From formulas (3.4) and (3.1) we can obtain easily

$$g_{j,k}(n+1)=\sum_{l=0}^{n}\binom{n}{l}f_{j,k}(l+1). \tag{3.6}$$

From (3.6) and (2.1) we can get

$$g_j(n+1)=\sum_{l=0}^{n}\binom{n}{l}f_j(l+1). \tag{3.7}$$

The inversion of (3.6) and (3.7) is of less importance and it follows from the same formulas as (3.6) and (3.7), by the simple derivation. We have

$$f_{j,k}(n+1)=\sum_{l=0}^{n}(-1)^{n-l}\binom{n}{l}g_{j,k}(l+1) \tag{3.8}$$

and

$$f_j(n+1)=\sum_{l=0}^{n}(-1)^{n-l}\binom{n}{l}g_j(l+1). \tag{3.9}$$

4. Generating functions

4.1. In the natural way we can define the generating function for $f_{j,k}(n)$ (j and k are fixed) as

$$F_{j,k}(t)=\sum_{n=1}^{\infty} f_{j,k}(n)\, t^n. \tag{4.1}$$

Substituting the result (3.2) in (4.1), we get

$$F_{j,k}(t)=\frac{t}{\pi}\int_{-\pi}^{\pi}\frac{\sin(j+1)x\sin(k+1)x}{1-2t\cos x}\,dx, \quad |t|<\frac{1}{2}. \tag{4.2}$$

By the aid of calculus of residues it follows

$$F_{j,k}(t)=\frac{u^{j+k+3}-u^{|j-k|+1}}{u^2-1} \quad \left(u=\frac{1-\sqrt{1-4t^2}}{2t}\right). \tag{4.3}$$

4.2. Similarly, for the generating function corresponding to $f_j(n)$ we have

$$F_j(t)=\sum_{n=1}^{\infty} f_j(n)\, t^n. \tag{4.4}$$

Using (2.1), (4.4) and (4.3) we can get

$$F_j(t)=-\frac{u^{j+2}-u}{(u-1)^2} \quad \left(u=\frac{1-\sqrt{1-4t^2}}{2t}\right). \tag{4.5}$$

4.3. If we define $G_{j,k}(t)$ and $G_j(t)$ in the same manner as $F_{j,k}(t)$ and $F_j(t)$, we can get from the definition of $G_{j,k}(t)$ and (3.5) the result corresponding to (4.2)

$$G_{j,k}(t)=\frac{t}{\pi}\int_{-\pi}^{\pi}\frac{\sin(j+1)x\sin(k+1)x}{1-t-2t\cos x}\,dx, \quad |t|<\frac{1}{3}. \tag{4.6}$$

From (4.6) it can be easily seen, that

$$G_{j,k}(t)=F_{j,k}\left(\frac{t}{1-t}\right), \tag{4.7}$$

and

$$G_j(t)=F_j\left(\frac{t}{1-t}\right). \tag{4.8}$$

5. Combinatorial identities

On the basis of direct comparison of the corresponding results (see (1.9) (3.3), (1.14), (3.35)), we can get the following identities:

$$\sum_{s=0}^{\left[\frac{j}{2}\right]}(-1)^s\binom{j-s}{s}\left\{\binom{n+j-2s}{(n+j-k-2s)/2}-\binom{n+j-2s}{(n+j-k-2s-2)/2}\right\} \tag{5.1}$$

$$=\binom{n}{(n+j-k)/2}-\binom{n}{(n+j+k+2)/2}, \quad \text{for} \quad n\equiv j+k \pmod 2,$$

$$(5.2)\qquad \sum_{s=0}^{j} c(-s-1, j-s)\{c(n+s, n+s-k)-c(n+s, n+s-k-2)\}$$
$$=c(n, n+j-k)-c(n, n+j+k+2).$$

In order to escape the condition with congruency in (5.1), we can modify (5.1) in the following way

$$(5.3)\qquad \sum_{s=0}^{\left[\frac{j}{2}\right]} (-1)^s \binom{j-s}{s} \left\{\binom{2(l-s)+k}{l-s}-\binom{2(l-s)+k}{l-s-1}\right\}=\binom{2l+k-j}{l}-\binom{2l+k-j}{l+k+1},$$

but then the similarity between (5.1) and (5.2) is lost.

REMARK. Some other identities can be obtained in the similar way but we shall not deal with them in the present paper.

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