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442. ON A GENERALISATION OF FAN-TODD'S INEQUALITY*

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In monograph [1], pp. 67—70, the following result due to OSTROWSKI is given:

Let $a=(a_1, \dots, a_n)$ and $b=(b_1, \dots, b_n)$ be real nonproportional sequences. Let $x=(x_1, \dots, x_n)$ be a real sequence such that

$$\sum_{i=1}^n a_i x_i = 0, \qquad \sum_{i=1}^n b_i x_i = 1.$$

Then

$$\sum_{i=1}^n x_i^2 \geqq \frac{\sum_{i=1}^n a_i^2}{\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n a_i^2\right)-\left(\sum_{i=1}^n a_i b_i\right)^2}$$

with equality if and only if

$$x_k = \frac{b_k \sum_{i=1}^n a_i^2 - a_k \sum_{i=1}^n b_i^2}{\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)-\left(\sum_{i=1}^n a_i b_i\right)^2} \qquad (k=1, \dots, n).$$

The corresponding inequality for complex numbers is also given in [1].

The above result may be presented in the following form (see: [2], p. 383):

Theorem 1. *Let a and b be linearly independent vectors in a unitary vector space V and let x be a vector in V such that*

$$(x, a)=0 \quad \textit{and} \quad (x, b)=1.$$

Then

$$G(a, b)\,\|x\|^2 \geqq \|a\|^2,$$

with equality if and only if

$$x=\frac{\|a\|^2 b-(b, a)a}{G(a, b)},$$

where $G(a, b)$ is Gram's determinant.

It is natural to try a generalisation of theorem 1 as follows:

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Theorem 2. *Let a and b be linearly independent vectors in a unitary vector space V and let x be a vector in V such that*

$$(x, a)=\alpha \quad \text{and} \quad (x, b)=\beta. \tag{1}$$

Then

$$G(a, b)\,\|x\|^2 \geqq \|\bar{\alpha} b-\bar{\beta} a\|^2, \tag{2}$$

with equality if and only if

$$x=\frac{(a, \bar{\beta} a-\bar{\alpha} b)\, b-(b, \bar{\beta} a-\bar{\alpha} b)\, a}{G(a, b)}. \tag{3}$$

Proof. Let y be the vector in V given by

$$G(a, b)\, y=(a, \bar{\beta} a-\bar{\alpha} b)\, b-(b, \bar{\beta} a-\bar{\alpha} b)\, a,$$

i.e.

$$G(a, b)\, y=\beta(a, a)\, b-\alpha(a, b)\, b-\beta(b, a)\, a+\alpha(b, b)\, a.$$

Then

$$G(a, b)\,(y, y)=\|\bar{\alpha} b-\bar{\beta} a\|^2 \tag{4}$$

and

$$G(a, b)\,(y, x)=\beta(a, a)\,(b, x)-\alpha(a, b)\,(b, x)-\beta(b, a)\,(a, x)+\alpha(b, b)\,(a, x).$$

Hence, in virtue of (1),

$$G(a, b)\,(y, x)=\|\bar{\alpha} b-\bar{\beta} a\|^2. \tag{5}$$

From (4) and (5) we get

$$(y, y)=(y, x). \tag{6}$$

Since

$$\|x-y\|^2 \geqq 0 \tag{7}$$

and using (6), we have

$$\|x\|^2 \geqq \|y\|^2.$$

Therefore, the inequality (2) is true.

According to (7), the equality holds in (2) if and only if $x=y$, i.e. if the condition (3) is satisfied.

The following result of FAN and TODD is also given in [1]:

Let $a=(a_1, \ldots, a_n)$ and $b=(b_1, \ldots, b_n)$ $(n \geqq 2)$ denote real sequences such that $a_i b_j \neq a_j b_i$ for $i \neq j$. Then

$$\frac{\sum_{i=1}^{n} a_i^2}{\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)-\left(\sum_{i=1}^{n} a_i b_i\right)^2} \leqq \binom{n}{2}^{-2} \sum_{\substack{i=1 \\ i \neq j}}^{n}\left(\sum_{j=1}^{n} \frac{a_j}{a_j b_i-a_i b_j}\right)^2.$$

The FAN-TODD's result may be presented in the following form:

Theorem 3. *Let* a *and* b *be vectors in a unitary vector space* V. *If* $\{a_1, \ldots, e_n\}$ *is an orthonormal basis of the space* V *and if* $(a, e_j)(b, e_i) \neq (a, e_i)(b, e_j)$ for $i \neq j$, *then*

$$\frac{\binom{n}{2}\|a\|^2}{G(a, b)} \leq \sum_{\substack{i=1 \\ i\neq j}}^{n} \left| \sum_{j=1}^{n} \frac{(a, e_j)}{(a, e_j)(b, e_i)-(a, e_i)(b, e_j)} \right|^2.$$

The theorem 3 can be generalised as follows.

Theorem 4. *Let* a *and* b *be vectors in a unitary vector space* V, *and let* α *and* β *be scalars. If* $\{e_1, \ldots, e_n\}$ *is an orthonormal basis of the space* V *and if* $(a, e_j)(b, e_i) \neq (a, e_i)(b, e_j)$ *for* $i \neq j$, *then*

(8) $$\binom{n}{2}^2 \frac{\|\bar{\beta} a-\bar{\alpha} b\|^2}{G(a, b)} \leq \sum_{\substack{i=1 \\ i\neq j}}^{n} \left| \sum_{j=1}^{n} \frac{(\beta a-\alpha b, e_j)}{(a, e_j)(b, e_i)-(a, e_i)(b, e_j)} \right|^2.$$

Proof. Let

(9) $$\binom{n}{2} x = \sum_{\substack{i=1 \\ i\neq j}}^{n} \left(\sum_{j=1}^{n} \frac{\beta(a, e_j)-\alpha(b, e_j)}{(a, e_j)(b, e_i)-(a, e_i)(b, e_j)} \right) e_i,$$

from which we deduce

(10) $$\binom{n}{2}(a, x) = \sum_{\substack{i=1 \\ i\neq j}}^{n} \left(\sum_{j=1}^{n} \frac{\beta(a, e_j)-\alpha(b, e_j)}{(a, e_j)(b, e_i)-(a, e_i)(b, e_j)} \right)(a, e_i)$$

$$= \beta \sum_{\substack{i=1 \\ i\neq j}}^{n} \left(\sum_{j=1}^{n} \frac{(a, e_j)(a, e_i)}{(a, e_j)(b, e_i)-(a, e_i)(b, e_j)} \right)$$

$$- \alpha \sum_{\substack{i=1 \\ i\neq j}}^{n} \left(\sum_{j=1}^{n} \frac{(b, e_j)(a, e_i)}{(a, e_j)(b, e_i)-(a, e_i)(b, e_j)} \right).$$

However, we have

$$\frac{(a, e_j)(a, e_i)}{(a, e_j)(b, e_i)-(a, e_i)(b, e_j)} + \frac{(a, e_i)(a, e_j)}{(a, e_i)(b, e_j)-(a, e_j)(b, e_i)} = 0$$

and

$$\frac{(b, e_j)(a, e_i)}{(a, e_j)(b, e_i)-(a, e_i)(b, e_j)} + \frac{(b, e_i)(a, e_j)}{(a, e_i)(b, e_j)-(a, e_j)(b, e_i)} = -1,$$

and so, (10) becomes

$$\binom{n}{2}(a, x) = \frac{n(n-1)}{2}\alpha.$$

Hence, we find

(11) $$(a, x) = \alpha.$$

In the same way we prove that

$$(b, x) = \beta. \tag{12}$$

Now, according to the theorem 2 and in view of (11) and (12), we get

$$\|x\|^2 \geqq \frac{\|\bar{\beta} a - \bar{\alpha} b\|^2}{G(a, b)}. \tag{13}$$

From (9) and (13) we obtain the inequality (8).

REFERENCES

1. D. S. Mitrinović: *Analytic Inequalities.* Berlin—Heidelberg—New York 1970.
2. S. Kurepa: *Konačno-dimenzionalni vektorski prostori i primjene.* Zagreb 1967.

COMMENT BY R. ASKEY

Theorem 2 is only a formal generalisation of the inequality of Fan and Todd. If one sets

$$x = Ay + Ba,$$

where $(y, a) = 0$ and $(y, b) = 1$, determines A and B by taking inner products

$$\alpha = (x, a) = A(y, a) + B(a, a) = B\|a\|^2,$$

$$\beta = (x, b) = A(x, b) + B(a, b) = A + \frac{\alpha}{\|a\|^2}(a, b),$$

uses

$$\|x\|^2 = |A|^2 \|y\|^2 + |B|^2 \|a\|^2, \text{ since } (y, a) = 0,$$

and then uses inequality of Fan-Todd on $\|y\|^2$, one obtains Theorem 2.