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418. A PROOF OF THE STEFFENSEN INEQUALITY*

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Theorem. *Let f and g be two continuous functions on $[a, b]$ such that f does not increase and that $0 \leqq g(x) \leqq 1$ for $x \in [a, b]$. Then we have*

$$(1) \qquad \int_{b-c}^{b} f(x)\,dx \leqq \int_{a}^{b} f(x)\,g(x)\,dx \leqq \int_{a}^{a+c} f(x)\,dx,$$

where $c = \int_a^b g(x)\,dx.$

Proof. First we suppose that f has the form

$$f(x) = y_i = \text{const} \qquad (x \in [x_i, x_{i+1});\ i = 0, 1, \ldots, n-1),$$

with $x_0 = a$, $x_n = b$ and $y_{i+1} \leqq y_i$ $(i = 0, 1, \ldots, n-1)$.

Since

$$c = \int_a^b g(x)\,dx \leqq b-a \quad \text{and} \quad c \geqq 0,$$

we have that $a \leqq b-c \leqq b$. Let $b-c \in [x_i, x_{i+1}]$. Then

$$(2) \qquad \int_{b-c}^{b} f(x)\,dx = y_i\,c + y_i\,(x_{i+1} - b) + \sum_{k=1}^{n-i-1} y_{i+k}\,(x_{i+k+1} - x_{i+k}).$$

From $g(x) \leqq 1$ and $y_i \geqq y_{i+k}$ $(k = 1, \ldots, n-i-1)$ we get

$$(y_i - y_{i+k})\,g(x) \leqq y_i - y_{i+k} \qquad (k = 1, \ldots, n-i-1),$$

whence it is

$$\int_{x_{i+k}}^{x_{i+k+1}} (y_i - y_{i+k})\,g(x)\,dx \leqq (y_i - y_{i+k})\,(x_{i+k+1} - x_{i+k}) \qquad (k = 1, \ldots, n-i-1).$$

* Presented February 5, 1973 by B. Crstici.

Adding the obtained inequalities we get

$$\sum_{k=1}^{n-i-1}(y_i-y_{i+k})\int_{x_{i+k}}^{x_{i+k+1}}g(x)\,dx\leqq y_i(b-x_{i+1})-\sum_{k=1}^{n-i-1}y_{i+k}(x_{i+k+1}-x_{i+k}).$$

The last inequality can be written in the form

$$\sum_{k=0}^{i}y_k\int_{x_k}^{x_{k+1}}g(x)\,dx+y_i\sum_{k=0}^{n-i-1}\int_{x_{i+k}}^{x_{i+k+1}}g(x)\,dx+y_i(x_{i+1}-b)$$

$$+\sum_{k=1}^{n-i-1}y_{i+k}(x_{i+k+1}-x_{i+k})\leqq\sum_{k=0}^{n-1}y_k\int_{x_k}^{x_{k+1}}g(x)\,dx.$$

Since $y_k\geqq y_i$ for $k\leqq i$, we find

$$y_i\sum_{k=0}^{i-1}\int_{x_k}^{x_{k+1}}g(x)\,dx+y_i(x_{i+1}-b)+\sum_{k=1}^{n-i-1}y_{i+k-1}(x_{i+k+1}-x_{i+k})\leqq\sum_{k=0}^{n-1}y_k\int_{x_k}^{x_{k+1}}g(x)\,dx.$$

Left side of this inequality is equal to the value of the integral $\int_{b-c}^{b}f(x)\,dx$ and the right side to the integral $\int_a^b f(x)g(x)\,dx$. Hence, the first inequality in (1) is proved. The second inequality in (1) will be proved by applying a procedure analogous to the above. First, we have

$$\int_a^{a+c}f(x)\,dx=y_ic+y_i(a-x_i)+\sum_{k=0}^{i-1}y_k(x_{k+1}-x_k).$$

Starting from $g(x)\leqq 1$ and $y_k\geqq y_i$ $(k=0,1,\ldots,i-1)$ we find

$$(y_k-y_i)g(x)\leqq y_k-y_i\qquad(k=0,1,\ldots,i-1),$$

from where we have

$$(y_k-y_i)\int_{x_k}^{x_{k+1}}g(x)\,dx\leqq(y_k-y_i)\,(x_{k+1}-x_k).$$

After addition we have

$$\sum_{k=0}^{i-1}(y_k-y_i)\int_{x_k}^{x_{k+1}}g(x)\,dx\leqq y_i(a-x_i)+\sum_{k=0}^{i-1}y_k(x_{k+1}-x_k).$$

This inequality can be put also in the form

$$\sum_{k=0}^{i-1} y_k \int_{x_k}^{x_{k+1}} g(x)\,dx + \sum_{k=0}^{n-i-1} y_{i+k} \int_{x_{i+k}}^{x_{i+k+1}} g(x)\,dx$$

$$\leqq y_i(a-x_i) + \sum_{k=0}^{i-1} y_k (x_{k+1}-x_k) + y_i \sum_{k=0}^{i-1} \int_{x_k}^{x_{k+1}} g(x)\,dx + \sum_{k=0}^{n-i-1} y_{i+k} \int_{x_{i+k}}^{x_{i+k+1}} g(x)\,dx.$$

Since $y_i > y_{i+k}$ $(k>0)$, this inequality gives

$$\sum_{k=0}^{n-1} y_k \int_{x_k}^{x_{k+1}} g(x)\,dx \leqq y_i\,(a-x_i) + \sum_{k=0}^{i-1} y_k\,(x_{k+1}-x_k) + y_i \int_a^b g(x)\,dx,$$

which is the second inequality from (1).

Inequality can be easily proved also when the function f is continuous and nonincreasing. The interval $[a, b]$ should be divided in n parts, and then the graph of f should be replaced by a polygonal line and finally allow n to tend to infinity.

In this way we shall get the inequality

$$\int_{b-c}^{b} f(x)\,dx \leqq \lim_{n\to+\infty} \sum_{i=0}^{n-1} f(x_i) \int_{x_i}^{x_{i+1}} g(x)\,dx \leqq \int_a^{b-c} f(x)\,dx.$$

Since

$$\lim_{n\to+\infty} \sum_{i=0}^{n-1} f(x_i) \int_{x_i}^{x_{i+1}} g(x)\,dx = \int_a^b f(x)\,g(x)\,dx,$$

the statement is proved in the general case.

If the function g is positive and greater than unity, but limited, then

$$\frac{g(x)}{M} = b(x) \leqq 1,$$

where M is an upper boundary of the function g in the interval $[a, b]$ and the given inequality has the form

$$\int_{b-c}^{b} f(x)\,dx \leqq \frac{1}{M} \int_a^b f(x)\,g(x)\,dx \leqq \int_a^{a+c} f(x)\,dx,$$

where

$$c = \frac{1}{M} \int_a^b g(x)\,dx.$$

But if the function g is negative in the interval $[a, b]$ and if its least value is $-m$, then the function $g(x)+m$ will be positive. If N is the upper boundary of the function $g(x)+m$ then we have

$$\int_{b-c}^{b} f(x)\,dx \leq \frac{1}{N}\int_{a}^{b} f(x)\big(g(x)+m\big)\,dx \leq \int_{a}^{a+c} f(x)\,dx,$$

where

$$c=\frac{1}{N}\int_{a}^{b}\big(g(x)+m\big)\,dx.$$

EDITORIAL NOTE

There is a number of proofs for STEFFENSEN'S inequality. See:

D. S. MITRINOVIĆ: *Analytic Inequalities.* Berlin-Heidelberg-New York 1970, pp. 107—119.

D. S. MITRINOVIĆ: *The Steffensen inequality.* These Publications № **247** — № **273** (1969), 1—14.

In these proofs even weaker assumptions of functions f and g are used (instead of being continuous, it is sufficient that f and g are integrable). The above proof is of interest, because it is directly connected to the definition of the integral.