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411. NUMERICAL ANALYSIS OF THE SOLUTION OF BESSEL'S DIFFERENTIAL EQUATION*

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1. INTRODUCTION

In the vast literature of BESSEL functions which represent solutions of the BESSEL differential equation

$$\frac{d^2 w}{dz^2}+\frac{1}{z}\,\frac{dw}{dz}+\left(1-\frac{v^2}{z^2}\right)w=0$$

we encounter two kinds of their elementary expansions. The first kind consists of convergent power series of BESSEL functions obtained by solving the BESSEL differential equation in the vicinity of the regular singularity $z=0$, where as the second kind of expansions are the asymptotic ones, obtained by HANKEL in solving the BESSEL differential equation in the vicinity of the irregular singularity $z \to \infty$. The convergent series are arranged by the positive powers of the argument z and they converge uniformly in the whole region, whereas the HANKEL expansions are arranged by the negative powers of that argument satisfying the POINCARÉ criterion [1]. The convergent series are practically applicable to the numerical calculations of BESSEL functions of small absolute value of the argument, while the asymptotic expansions become applicable for large values of the argument. On the other hand the range ot practical applicability of convergent series is becoming wider for larger values of the order ν, whereas the region of apparent convergence of asymptotic expansions is getting narrower. However, apart form all these facts there is a region in which none of these series is practically applicable. This region is known in literature as *the region of large order and large argument*, but a more definite definition for it was not given so far.

With the development and application of new methods and procedures, the polynomial asymptotic expansions have been developed which facilitate numerical calculations of BESSEL functions of positive values of the order ν

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and of the argument x in the region of large order and large argument. However, the region of their application is not strictly defined. We distinguish three types of expansions with respect to the ratio x/ν: $x/\nu>1$, $x/\nu<1$ and $x/\nu=1$.

MEISSEL has obtained the asymptotic expansions of the BESSEL functions for all three cases [2] in an elementary way, using a convenient substitution of variables in the BESSEL differential equation. By developing and applying the method of the steepest descents in the theory of contour integrals [3], DEBYE has designed the adequate contours for the given cases starting form the SOMMERFELD's integrals of HANKEL functions, and by a convenient substitution of variables he has obtained the asymptatic expansions of BESSEL functions for real [3] and complex values of the order and argument [4].

The investigation of MEISSEL's expansions shows that they are convenient for numerical calculations of $J_\nu(x)$ and $Y_\nu(x)$ in the case of $x=\nu+O(\nu^{1/3})$, while the DEBYE expansions appear to be applicable if $|x-\nu|$ is large with respect to $\nu^{1/3}$. Thus there appears a region of transition when $|x-\nu|$ is comparable to $\nu^{1/3}$, where none of these expansions is applicable [5].

NICHOLSON has obtained asymptotic expansions applying KELVIN's principle of the stationary phase. These expansions satisfy the region of transition of MEISSEL's and DEBYE's expansions, but they cannot be expressed by elementary functions [2. p. 248].

F. W. J. OLVER has obtained asymptotic expansions of the HANKEL functions $H_\nu^{(1)}(\nu+\tau\nu^{1/3})$ and $H_\nu^{(2)}(\nu+\tau\nu^{1/3})$ for large ν, where τ is a constant, applying "the method of the steepest descents". These expansions satisfy the above region and they are very convenient for the analysis of the zeros of the BESSEL functions $J_\nu(x)$ and $Y_\nu(x)$.

Besides the recent papers of OLVER [6, 7] it is worth mentioning C. S. MEIJER [8], and W. G. BICKLEY [9], who succeeded by an elementary approach to obtain the asymptotic expansions for real values of the order and argument, which are very convenient for practical calculations [10]. However, all these expansions are polynomial. They are given with a finite number of terms which cannot be generated recursively. The regions of their application are not strictly defined and therefore, besides tabular constructions, they did not find practical application in the automatic control of the calculation processes using modern digital computers.

By summation of alternative series on a finite hardware format of a digital computer an error is induced which is proportional to the difference of the powers of the absolute values of the maximal term and the sum of the summed series. This error is known as the "round-off error" but it could be called also the "error of numerical truncation". The question of the maximal term is directly connected with the convergence speed of the convergent series, and with the apparent convergence speed of the asymptotic expansions respectively. Thus the problem of the region of large order and large argument can be stated from the numerical truncation error aspect as well as from the aspect of analytical truncation error, appearing if the first omitted term is not negligible with respect to the sum of previous terms on the given hardware format of the digital computer. The error of analytical truncation may be relevant for all asymptotic expasions, because the summation of their terms is limited by the summing of the term of minimal absolute value [1].

These are the main reasons for the practical inapplicability of the existing series and expansions for generating the BESSEL functions on digital computers by automatic control of the calculation processes in an arbitrary region of the order and the argument, and they have hereby caused the emergence of new procedures. However, these procedures have shown their own disadvantages. The procedures of classical interpolation by orthogonal polynomials [**11**, **12**, **13**, **14**, **15**], the phase — amplitude procedure [**15**, **16**, **17**], and the quadrature procedure [**18**, **19**] have the same or similar limitations [**20**, **21**, **22**]. A strict application of the recursive techniques [**23**. pp. XVII, **21**, **24**, **25**, **26**] requires an iterative procedure of generating a set of functions in order to test the necessary choice of the position of the initial functions or the tabular orientation of these positions. In the former case it is possible to satisfy an arbitrarily given error on account of the procedure speed, while in the latter, the error is satisfied in the predetermined region of the arguments (ν, x).

By stating the criterion of the maximal summand in the analysis of numerical truncation error and by analysing the term of minimal absolute value in the estimation of analytical truncation error in summing the HANKEL asymptotic expansions, the separation of the region of favourable convergence of convdrgent series, from the apparent convergence of the HANKEL expansions has been performed and the region of large order and large argument (ν, x) has been defined in this way. By the analysis of the absolute values of the maximal terms of the convergent series and the HANKEL expansions of BESSEL functions it is shown that the curves of equivalent values of these terms are at the same time the curves of equivalent indices, and also that they converge asymptotically to the second order paraboles. The first of these curves are paraboles in x, and the others are paraboles in ν. However, in both cases the absolute values of maximal terms can be asymptotically represented by a unique function

$$\varphi(\zeta) = \frac{e^{\zeta}}{\sqrt{2\pi\zeta}},$$

where ζ is the index of the maximal term. On the other hand, it is shown that the curve of mutual cross sections of the corresponding curves of these two families converges asymptotically in an oscillatory manner into a straight line with very small initial amplitudes. Therefore, if the point (ν, x) is below this line and the value of the function $\varphi(\zeta)$ exceeds the one corresponding to the permitted absolute error for a given index of the maximal term of a convergent series, or if the point (ν, x) is above the line and the value of the given function is greater than the value corresponding to the permitted absolute error for a given index of the maximal term of the HANKEL expansions, one can say that the point (ν, x) lies in the region of large order and large argument.

Starting from these regularities it is possible to speed up considerably the procedure of generating any kind of BESSEL's functions in the whole region of the arguments (ν, x), by applying elementary series and expansions for direct calculation outside the region of large order and large argument and by modification of the recursive procedure for the calculation of the functions in the region. This may be of special interest for generating the BESSEL functions of a complex order and argument too. The analysis presented here is concerned with real values ot the order and the argument.

2. ANALYSIS OF THE TERMS OF ELEMENTARY SERIES WITH EXTREME ABSOLUTE VALUES

Let us consider the normalized series of BESSEL's functions of the first kind (of order ν)

$$E_\nu(x)=\sum_{m=0}^{\infty}(-)^m\frac{(x/2)^{2m}}{m!\,(\nu+1)_m},$$

where

$$(\nu+1)_m=\frac{\Gamma(\nu+m+1)}{\Gamma(\nu+1)}$$

and the HANKEL asymptotic expansions of the modified BESSEL functions of the second kind

$$F_\nu(x)\sim\sum_{n=0}^{\infty}\frac{(\nu,n)}{(2x)^n}$$

where (ν, n) is HANKEL's symbol:

$$(\nu,n)\equiv\frac{(4\nu^2-1^2)(4\nu^2-3^2)\cdots[4\nu^2-(2n-1)^2]}{2^{2n}\,n!},\qquad (\nu,0)\equiv 1,$$

for real and positive values of the arguments ν and x.

The functional series $E_\nu(x)$ converges uniformly in the whole region of the agrument x. Depending on the relation between the agruments ν and x,

$$(x/2)^2>m(\nu+m),$$

its terms increase in absolute value up to the term of maximum absolute value of index M,

$$M=\left[\frac{1}{2}\left(\sqrt{x^2+\nu^2}-\nu\right)\right]\leq\frac{1}{2}\left(\sqrt{x^2+\nu^2}-\nu\right)=\mu$$

and then they decrease monotonically.

The functional series $F_\nu(x)$ converges asymptotically in the region of the agrument x.

Its terms increase up to

$$4\nu^2-(2n-1)^2>8nx$$

i. e., up to the maximal term, of index Λ,

$$\Lambda=\left[\sqrt{\left(x-\frac{1}{2}\right)^2+\nu^2-\frac{1}{4}}-\left(x-\frac{1}{2}\right)\right]\leq\sqrt{\left(x-\frac{1}{2}\right)^2+\nu^2-\frac{1}{4}}-\left(x-\frac{1}{2}\right)=\lambda,$$

and then, under the condition that $\nu\neq n-\frac{1}{2}$, they decrease to

$$4\nu^2-(2n-1)^2>-8nx,$$

i.e., to the term of minimal absolute value, of index x,

$$k=\left[\sqrt{\left(x+\frac{1}{2}\right)^2+\nu^2-\frac{1}{4}}+\left(x+\frac{1}{2}\right)\right]\leq\sqrt{\left(x+\frac{1}{2}\right)^2+\nu^2-\frac{1}{4}}+\left(x+\frac{1}{2}\right)=\varkappa$$

and then they increase without a limit.

In the case of $\nu = n - \frac{1}{2}$, the asymptotic series is interrupted and it transforms into a finite sum. The cylindric functions transform into spherical ones, expressed by elementary functions.

In the opposite case, for $n > \nu + \frac{1}{2}$, the asymptotic series $F_\nu(x)$ becomes alternative and therefrom the previous condition of the negative slope of its terms follows.

For $n < \nu + \frac{1}{2}$, the terms of the asymptotic series $F_\nu(x)$ are positive and there is no problem in their summation. However, these terms form alternative series of BESSEL's functions of the first and second kinds, $J_\nu(x)$ and $Y_\nu(x)$, thus the problem of their summation in the region of apparent convergence, to the minimal term, is adequate for the problem of summation of convergent series of the given functions.

By summing up the term of maximal absolute value and the numeric truncation error is induced whose level canot be reduced by further summation to the term of negligible absolute value.

On the other hand, the term of minimal absolute value of the asymptotic series defines the region of their application from the analytical trucation error point of view.

In the following text we shall understand under the extreme values the extreme absolute values.

2.1. Analysis of the maximal terms of the convergent series $E_\nu(x)$

Let us consider the function of the maximal term of the series $E_\nu(x)$ as a function of continuous values of the index μ, (§ 2),

$$g(\nu, x) = \frac{\Gamma\,(x/2)^{2\mu}\,\Gamma(\nu+1)}{\Gamma(\mu+1)\,\Gamma(\nu+\mu+1)}.$$

Substituting $\alpha = x/\nu$, we obtain

$$\mu = \frac{\nu}{2}\left(\sqrt{1+\alpha^2} - 1\right)$$

and by the further substitution of

$$\alpha = \frac{2\eta}{1-\eta^2},$$

we obtain

$$\mu = \nu\,\frac{\eta^2}{1-\eta^2}.$$

Let us consider now sufficiently large values of the agruments ν and x, so that we can apply STIRLING's asymptotic formula for the Gamma-functions,

$$\Gamma(z+1) \sim \sqrt{2\pi z}\,(z/e)^z\left\{1 + O\left(\frac{1}{z}\right)\right\}.$$

Applying these relations, after a short rearrangement, we obtain an asymptotic expression for the function of the maximal term in the form:

$$g(\nu, \eta; \mu) \sim \sqrt{\frac{1-\eta^2}{2\pi\mu}}\, e^{2\mu} \left(\frac{\nu}{\nu+\mu}\right)^{\nu} \left\{1 + O\left(\frac{1}{\nu}, \frac{1}{\mu}, \frac{1}{\nu+\mu}\right)\right\}.$$

Let us consider now the locus of equivalent values of the function $g(\nu, \eta; \mu)$, i. e.

$$g(\nu, \eta; \mu) = C_\mu.$$

For the existence of curves for any values of ν, and foı $\nu \to +\infty$ too, there must obviously exist the limiting values

$$\lim_{\nu \to +\infty} \eta = 0, \qquad \lim_{\nu \to +\infty} \mu = \mu_0.$$

Consequently, the limit of the asymptotic functions $g(\nu, \eta; \mu)$, when $\nu \to +\infty$, can be expressed by the function

$$\varphi(\mu_0) = \frac{e^{\mu_0}}{\sqrt{2\pi\mu_0}},$$

and the asymptotic forms of the curves of equivalent maximal terms by paraboles of the second order

$$\nu \sim \frac{1}{\mu_0}\left\{\left(\frac{x}{2}\right)^2 - \mu_0{}^2\right\},$$

as follows immediately from the definition of the maximal term.

2.2. Analysis of the maximal terms of the asymptotic expansion $F_\nu(x)$

Let us consider, as previously, the function of the maximal term of the series $F_\nu(x)$ as a function of continuous values of the index λ, (§2),

$$h(\nu, x) = \frac{1}{(2x)^\lambda\, \Gamma(\lambda+1)} \frac{\Gamma\left(\nu+\lambda+\frac{1}{2}\right)}{\Gamma\left(\nu-\lambda+\frac{1}{2}\right)}.$$

Considering sufficiently large arguments ν and x such that $\nu, x \gg \frac{1}{2}$, we can write:

$$\lambda \simeq \nu\left(\sqrt{1+\alpha^2} - \alpha\right),$$

where α, as in § 2.1, is

$$\alpha = \frac{x}{\nu} \quad \text{and} \quad \alpha = \frac{2\eta}{1-\eta^2}$$

respectively, so that $\lambda \simeq \nu \frac{1-\eta}{1+\eta}$.

Applying STIRLING's formula for sufficiently large values of the arguments ν and x xnd rearranging the expression of the function of the maximal term $h(\nu, x)$, we obtain its asymptotic form:

$$h(\nu, \lambda) \sim \frac{1}{\sqrt{2\pi\lambda}} e^{-\lambda} \left(\frac{\nu+\lambda}{\nu-\lambda}\right)^{\nu} \left\{1 + O\left(\frac{1}{\lambda}, \frac{1}{\nu+\lambda-\frac{1}{2}}, \frac{1}{\nu-\lambda-\frac{1}{2}}\right)\right\}.$$

The condition for the existence of the curves of equivalent values of the function $h(\nu, \lambda)$,

$$h(\nu, \lambda) = C_\lambda$$

for any value ν, including $\nu \to +\infty$, are the following limits

$$\lim_{\nu \to +\infty} \eta = 1, \quad \lim_{\nu \to +\infty} \lambda = \lambda_0.$$

Consequently, we arrive at the same conclusion as in the previous section: along the lines of equivalent maximal terms, the index of the maximal term approaches a limiting value, and the function of the maximal term can be expressed in the form

$$\varphi(\lambda_0) = \frac{e^{\lambda_0}}{\sqrt{2\pi\lambda_0}}$$

where in both cases μ_0 and λ_0 are the limiting values of the indices of the maximal terms of the corresponding series. The asymptotic curves of the equivalent maximal terms of the asymptotic expansions can be obtained in a similar way as the asymptotic curves of the equivalent maximal terms of the convergent series from the definition of the maximal term under the condition of the existence of its limiting value, λ_0,

$$x \sim \frac{1}{2\lambda_0}(\nu^2 - \lambda_0^2).$$

2.2.1. *Analysis of the minimal terms of the asymptotic expansions*

Using the relations for the indices of the maximal and minimal terms of the asymptotic series $F_\nu(x)$, λ and $\varkappa$, §2, it is easy to obtain the inverse relations for the arguments x and ν:

$$x = \frac{\varkappa-\lambda}{2}\left(1 - \frac{1}{\varkappa+\lambda}\right) \quad \text{and} \quad \nu^2 = \varkappa\lambda\left(1 - \frac{2}{\varkappa+\lambda}\right) + \frac{1}{4}.$$

For sufficiently large values of the arguments x and ν, such that $x, \nu \gg 1/2$, it follows that:

$$\lambda \simeq \sqrt{x^2+\nu^2} - x \quad \text{and} \quad \varkappa \simeq \sqrt{x^2+\nu^2} + x,$$

or

$$x \simeq \frac{\varkappa-\lambda}{2} \quad \text{and} \quad \nu^2 = \varkappa\lambda$$

respectively.

Contrary to the previous analysis, where the index of the maximal term was $\lambda<\nu$, we have the opposite case here. The index of the minimal term is larger than the order of the function and for a given λ it increases with the second power of the order. Consequently it is convenient to consider the behaviour of the minimal term along the curve of equivalent maximal terms ($\lambda=$const).

Applyng the relation between Gamma-functions and the sine function, we can obtain the function of the minimal term of the asymptotic expansion $F_\nu(x)$, for continuous values of the index $\varkappa$, (§2), in the form

$$F(\nu, x)=\frac{\cos[\pi(\varkappa-\nu)]}{\pi(2x)^{\varkappa}\Gamma(\varkappa+1)}\Gamma\left(\varkappa+\nu+\frac{1}{2}\right)\Gamma\left(\varkappa-\nu+\frac{1}{2}\right).$$

Applying STIRLING's formula, and after some rearrangement, one obtains the asymptotic form of the function of the minimal term:

$$f(\nu, \varkappa)\sim\sqrt{\frac{2}{\pi\varkappa}}\cos[\pi(\varkappa-\nu)]\,e^{-\varkappa}\left(\frac{\varkappa+\nu}{\varkappa-\nu}\right)^{\nu}\{1+O_\nu(\varkappa)\}.$$

For sufficiently large values of the order ν, along the curves of equivalent maximal terms, ($\lambda=$const) substituting $\nu^2=\varkappa\lambda$, one obtains:

$$f(\lambda, \varkappa)\sim\sqrt{\frac{2}{\pi\varkappa}}\cos[\pi(\varkappa-\nu)]\,e^{2\lambda-\varkappa}.$$

The minimal term is a function of integer values of the arguments λ and $\varkappa$ and it reduces to zero in the case of $\nu=[\varkappa]-\frac{1}{2}$.

The amplitudes of the previous function may be of interest to us as predictions for the analytic truncation error in the application of the asymptotic expansions to numerical computations of BESSEL's functions. Introducing the argument x we can write

$$f_0(x, \lambda)\simeq\sqrt{\frac{2}{\pi(2x+\lambda)}}\,e^{-2x+\lambda},$$

where λ is the index of the maximal term.

2.3. Locus of the equal maximal terms of the convergent series $E_\nu(x)$ and the asymptotic expansions $F_\nu(x)$

Comparing the general asymptotic expressions for functions of maximal terms of convergent series (§ 2.1) and of asymptotic expansions (§ 2.2.), we obtain an asymptotic curve of mutually equal absolute values of maximal terms of convergent series and HANKEL's asymptotic expansions

$$\sqrt{\frac{1-\eta^2}{2\pi\mu}}\,e^{2\mu}\left(\frac{\nu}{\nu+\mu}\right)^{\nu}\left\{1+O_\mu\left(\frac{1}{\nu}\right)\right\}\sim\frac{1}{\sqrt{2\pi\lambda}}\,e^{-\lambda}\left(\frac{\nu+\lambda}{\nu-\lambda}\right)\left\{1+O_\lambda\left(\frac{1}{\nu}\right)\right\},$$

where μ, λ and η are variables defined by the relations in §§ 2.—2.2, and ν is the order of the BESSEL functions, and the argument of the series $E_\nu(x)$ and $F_\nu(x)$ respectively.

Eliminating μ and λ we can write the previous relations in the form:

$$\eta(1-\eta^2)\,e^{\frac{3\eta^2-2\eta+1}{1-\eta^2}} \sim \left\{\frac{\eta}{1-\eta^2}\left[1+O_\eta\left(\frac{1}{\nu}\right)\right]\right\}^{1/\nu}.$$

If we assume that the indices of the maximal terms along the curves of their equivalent values are constant for both kinds of series, as it is shown by the analysis of the asymptotic region (§§ 2.1—2.2), then we can expect that η varies from 1 to 0 for the case of the convergent series, and from 0 to 1 for the case of the asymptotic expansions when ν takes the values from 0 to infinity respectively. Consequently, the intersections of the curves of these two families can be expected in the interval $\eta \in (0, 1)$.

On the other hand, having in mind that the asymptotic functions of the maximal terms of both series are functions of the same form and the same arguments (§§ 2.1—2.2), we can reduce the mutual equality of the maximal terms of these series to the equality of relevant arguments, i.e. of their indices μ and λ (§§ 2.1—2.2):

$$\nu\frac{\eta^2}{1-\eta^2}=\nu\frac{1-\eta}{1+\eta},$$

implying $\eta = 1/2$.

Although the result is approximate, as it is concerned with mutual intersections of the considered families of curves in the region adjacent to the asymptotic one, it points to the existence of the solution $\eta \neq 0.1$.

Therefore, the asymptotic locus of equal maximal terms of the convergent series $E_\nu(x)$ and the asymptotic expansions $F_\nu(x)$ is a straight line

$$x = \alpha\nu + \beta,$$

with the slope $\alpha = \frac{2\eta}{1-\eta^2}$, where η is a solution of the transcedent equation

$$\eta(1-\eta^2)\,e^{\frac{3\eta^2-2\eta+1}{1-\eta^2}} = 1.$$

Solving this equation we obtain $\eta = 0.494202620730667\ldots$, which is in a good agreement with the approximative solution $\eta = 1/2$. From this follows the conclusion about treating of asymptotic relations (§§ 2.1, 2.2) as approximative relations with a sufficient degree of accuracy, which is of special importance from the point of view of this work.

Substituting the value for η we obtain $\alpha = 1.30782300122980\ldots$

3. NUMERICAL CHECKING OF THE RESULTS

For a prescribed value of the order $\nu\,(=\nu_0)$, and under the condition that the maximal terms of the convergent series $E_\nu(x)$ and the asymptotic expansions $E_\nu(x)$ are equal, the point (ν, x) on the curve § 2.3 is determined by iteration. Starting from this point, for the determined value of the maximal terms, the curves of equivalent values of the maximal terms of the mentioned series are drawn in the same way (§§ 2.1, 2.2). Simultaneously, along the curves of equivalent maximal terms of the asymptotic expansions $F_\nu(x)$ the corresponding values of the minimal terms are found. The procedure has been repeated in several subintervals of the interval $\nu \in [2,1500]$ with the corresponding increment $\Delta\nu$.

On the basis of the obtained results the following conclusions have been drawn:

(1) From the points of mutual intersections, the curves of equivalent maximal terms of the series $E_\nu(x)$ and $F_\nu(x)$ approximate well the corresponding paraboles (§§ 2.1—2.2), and the indices of the maximal terms converge asymptotically to a constant. With few exceptions for this region, integer values $M=[\mu]$ and $\Lambda=[\lambda]$ along these curves can be regarded as constants.

(2) The amplitudes of the minimal terms fall sharply along the curve § 2.2, while the function to (§ 2.2.1) represents a prediction for the exact order of magnitude of these terms. For the points of mutual intersections (§ 2.3) the order of magnitude of these terms is $10^{-\nu}$.

(3) The locus of mutually equal maximal terms of the convergent series $E_\nu(x)$ and the asymptotic expansions $F_\nu(x)$, (§ 2.3), can be represented by a damped periodic curve with a decreasing period, and an oscillation axis

$$x = 1.3078230012\nu + 0.155\overline{1},$$

which, consequently, is the asymptote of the curve § 2.3.

The maximal amplitude of this periodic curve, with respect to the asymptote, is of the order 10^{-2}. Its positive slopes are considerably more marked as a consequence of the discrete values of the index of the maximal terms.

4. DEFINITION OF THE REGION OF LARGE ORDER AND LARGE ARGUMENT

On the basis of the presented analysis which is based on the mutual dependence of the absolute values of the maximal terms. the speed of convergence of the corresponding series and the error of numerical truncation reduced during their generation, one can define the region of large order ν and large argument x of BESSEL's functions as follows:

— If the point (ν, x) lies under the asymptote of the locus of the mutually equal maximal term of the convergent series and HANKEL's asymptotic expansions of BESSEL's function

$$x = \alpha\nu + \beta, \qquad (\S\ 2.3),$$

and if the index of the maximal term of the convergent series

$$M = \left[\frac{1}{2}\left(\sqrt{x^2+\nu^2} - \nu\right)\right]$$

exceeds the value for which the maximal term corresponds to the permitted absolute error, or if the point (ν, x) lies above the asymptote and the index of the maximal term of HANKEL's expansion

$$\lambda = \left[\sqrt{x^2+\nu^2} - x\right],$$

exceed the value for which the maximal term corresponds to the permitted absolute error, respectivelly, one can say that the point (ν, x) is in the region of large order and large argument. There is no influence of the term of minimal absolute value of HANKEL's asympotic expansions in the region of large order and large argument, (§ 3).

The families of the curves of equivalent values $C_m = e^m/\sqrt{2\pi m}$ of the maximal terms of convergent series and HANKEL's asymptotic expansions, the asymptote of their mutual intersections, the position of the points (ν, x) in the

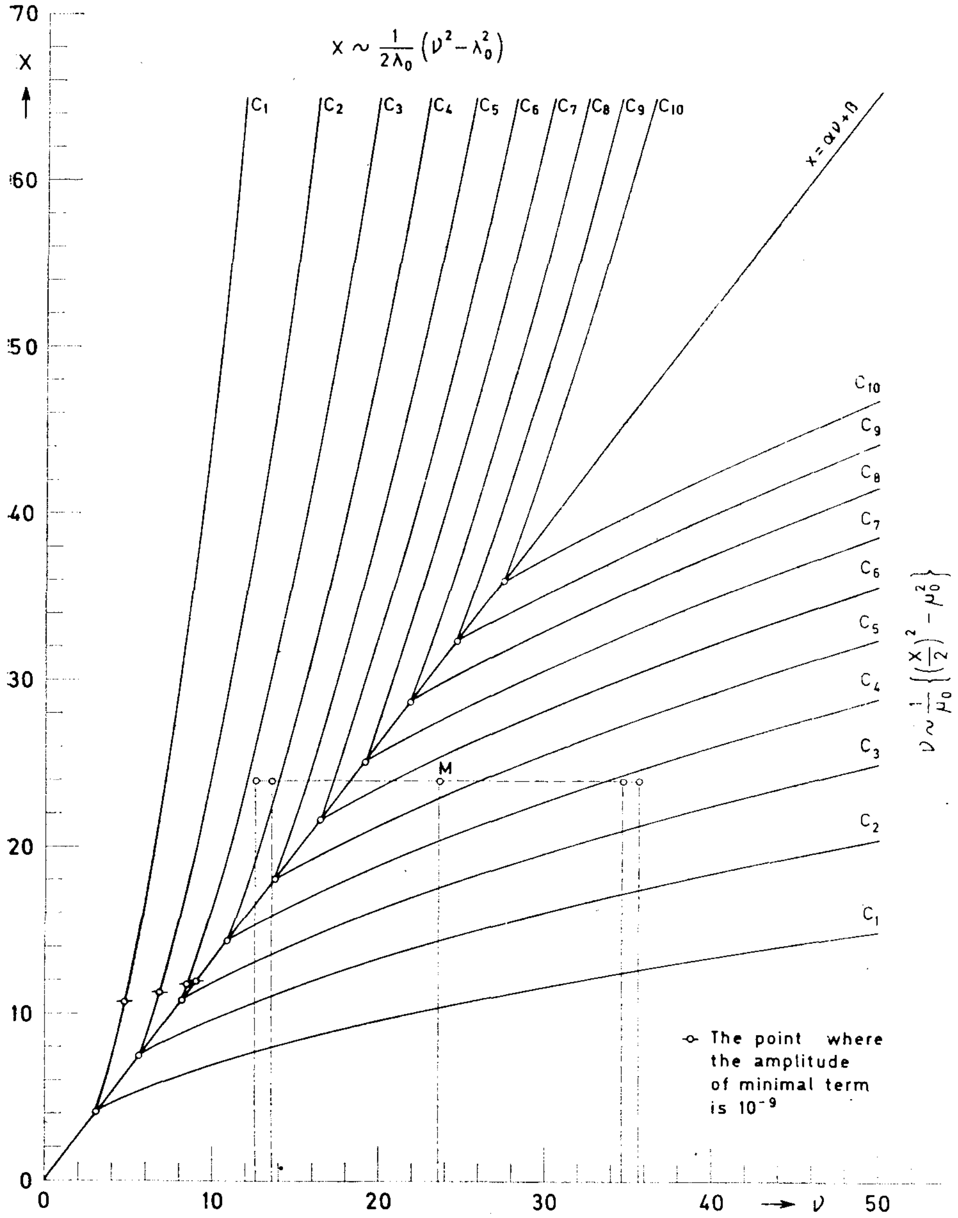

Fig. 1

region of asymptotic expansions for which the amplitudes of the minimal terms are of the order 10^{-9} as well as, the translations of the point M from the region of large order and large argument if this region is defined by the curves C_4 are shown on Fig. 1. The maximal term of these curves correspond to an error of numerical truncation of the order 10^{-9} on a hardware format of 36 binary digits.

The results of our analysis are applied to develop a programme for generating BESSEL's functions of all four forms in the region of real arguments (ν, x). The results obtained confirmed the validity of the definition of the region of large order and large argument from the point of view of application of the elementary series for numerical computations outside the region and for a modification of the recursive procedure within the region.

The numerical checking of the expounded analysis and the programme for generating the BESSEL functions were performed on CDC 3600 computer at the Institute „Boris Kidrič" — Vinča.

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