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383. A REMARK ON THE SCHWEITZER AND KANTOROVICH INEQUALITIES*

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In 1914 P. SCHWEITZER proved the following theorem: If $\{a_1, \dots, a_n\}$ is a set of real numbers with the property

$$0<m \leqq a_k \leqq M \qquad (k=1, \dots, n),$$

then

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}\right) \leqq \frac{(M+m)^2}{4Mm}. \tag{1}$$

In [4] (see also [3]) G. PÓLYA and G. SZEGÖ have obtained the following generalization: If $\{A_1, \dots, A_n\}$, $\{B_1, \dots, B_n\}$ are real numbers which verify

$$0<m_1 \leqq A_k \leqq M_1, \qquad 0<m_2 \leqq B_k \leqq M_2 \qquad (k=1, \dots, n),$$

then

$$\frac{\left(\sum_{k=1}^{n} A_k^2\right)\left(\sum_{k=1}^{n} B_k^2\right)}{\left(\sum_{k=1}^{n} A_k B_k\right)^2} \leqq \left(\frac{\sqrt{\frac{M_1 M_2}{m_1 m_2}}+\sqrt{\frac{m_1 m_2}{M_1 M_2}}}{2}\right)^2. \tag{2}$$

Then they showed that the equality is attained in (2) if and only if the following conditions are simultaneously fulfilled:

I. $$\alpha := \frac{\frac{M_1}{m_1}}{\frac{M_1}{m_1}+\frac{M_2}{m_2}}\, n, \qquad \beta := \frac{\frac{M_2}{m_2}}{\frac{M_1}{m_1}+\frac{M_2}{m_2}}\, n$$

are natural numbers.

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II. α of the numbers $A_1, \dots, A_n$ are equal to m_1, and β of these numbers equal to M_1, and if the corresponding numbers B_k are equal to M_2 and m_2 respectively.

If $A_k := \sqrt{a_k}$, $B_k := \frac{1}{\sqrt{a_k}}$, $M_1 := \sqrt{M}$, $m_1 := \sqrt{m}$, $M_2 := \frac{1}{\sqrt{m}}$, $m_2 := \frac{1}{\sqrt{M}}$, then (2) is the same inequality as (1). On the other hand in this case

$$\alpha = \beta = \frac{n}{2}$$

such that for $n = 2p+1$ the SCHWEITZER's result does not give the best bound, taking into account that in (1) the equality is not attained (except some trivial situations as: $m = a_k = M = 1$, $k = 1, \dots, n$).

The object of this note is to improve SCHWEITZER's inequality.

Theorem 1. *If* $\{a_1, \dots, a_n\}$ *is a set of real numbers with*

$$0 < m \leqq a_k \leqq M \qquad (k = 1, \dots n),$$

then

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \leqq \frac{\left(\left[\frac{n}{2}\right] M + \left[\frac{n+1}{2}\right] m\right)\left(\left[\frac{n+1}{2}\right] M + \left[\frac{n}{2}\right] m\right)}{Mm}$$

where the symbol $[\cdot]$ *denotes the integral part.*

The equality is attained if $\left[\frac{n}{2}\right]$ *of the numbers* $a_1, \dots, a_n$ *are equal to* m *(or to* M*) and if* $\left[\frac{n+1}{2}\right]$ *of these numbers are equal to* M *(or to* m*).*

Proof. It is stated in [1] (see also 3.2.26, page 205 in [3]) that if $\{A_1, \dots, A_n\}$, $\{B_1, \dots, B_n\}$ are real numbers with

$$m_1 \leqq A_k \leqq M_1, \qquad m_2 \leqq B_k \leqq M_2 \qquad (k = 1, \dots, n),$$

then

$$\left| \sum_{k=1}^{n} A_k B_k - \frac{1}{n} \sum_{k=1}^{n} A_k \sum_{k=1}^{n} B_k \right| \leqq (M_1 - m_1)(M_2 - m_2)\, w(n)$$

where

$$w(n) = \begin{cases} \frac{n}{4} & \text{for } n \text{ even,} \\ \frac{n^2-1}{4n} & \text{for } n \text{ odd.} \end{cases}$$

If we take

$$A_k := a_k > 0, \quad B_k := \frac{1}{a_k}, \quad M_1 := M, \quad m_1 = m, \quad M_2 := \frac{1}{m}, \quad m_2 := \frac{1}{M},$$

then from the above inequality we obtain

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) \leqq \frac{(M+m)^2}{4Mm} n^2 - g(n)$$

where

$$g(n)=\begin{cases} 0, & \text{for } n \text{ even} \\ \dfrac{(M-m)^2}{4Mm} & \text{for } n \text{ odd.} \end{cases}$$

An easy computation shows that

$$\frac{(M+m)^2}{4Mm}n^2-g(n)=\frac{\left(\left[\frac{n}{2}\right]M+\left[\frac{n+1}{2}\right]m\right)\left(\left[\frac{n+1}{2}\right]M+\left[\frac{n}{2}\right]m\right)}{Mm}$$

and the theorem is proved.

In [2] P. HENRICI showed that the KANTOROVICH's inequality

$$\left(\sum_{k=1}^{n} a_k p_k\right)\left(\sum_{k=1}^{n}\frac{1}{a_k}p_k\right)\leqq\frac{(M+m)^2}{4Mm}$$

$$0<m\leqq a_k\leqq M, \quad p_k\geqq 0, \quad \sum_{k=1}^{n} p_k=1 \quad (k=1,\ldots,n),$$

is a simple consequence of SCHWEITZER's inequality (1).

By using the same method of proof which is described in [2], from the above theorem we obtain the following improvement of KANTOROVICH's result.

Theorem 2. *Let* $\{p_1,\ldots,p_n\}$ *and* $\{a_1,\ldots,a_n\}$ *be real numbers which verify*

$$p_k\geqq 0, \quad \sum_{k=1}^{n} p_k=1, \quad 0<m\leqq a_k\leqq M \quad (k=1,\ldots,n).$$

Then

$$\left(\sum_{k=1}^{n} a_k p_k\right)\left(\sum_{k=1}^{n}\frac{1}{a_k}p_k\right)\leqq\frac{\left(\left[\frac{n}{2}\right]M+\left[\frac{n+1}{2}\right]m\right)\left(\left[\frac{n+1}{2}\right]M+\left[\frac{n}{2}\right]m\right)}{Mmn^2}.$$

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