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363. SOME INEQUALITIES FOR TRIANGLES*

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1. Inequalities are obtained for pairs of triangles, plane or spherical, which are such that the angles (sides) of one triangle lie between the least and greatest of the angles (sides) of the other, the two triangles having equal angle (side) sum. The results follow from Theorem 108 in the famous treatise *Inequalities* by HARDY, LITTLEWOOD and POLYÁ (Cambridge, 1934). (I am indebted to PETAR M. VASIĆ for pointing out that the theorem I had originally used to obtain these inequalities was simply a special case of Theorem 108.)

Inequalities are also found for pairs of triangles whose sides satisfy the conditions $a'\geqq a$, $b'\geqq b$, $c'\geqq c$: for triangles such that $a=\max(a_1, a_2)$, $b=\max(b_1, b_2)$, $c=\max(c_1, c_2)$: for triangles such that $a=(a_1^p+a_2^p)^{1/p}$ $(p=1, 2, 4)$, etc.

2. I quote (with appropriate expansion) from *Inequalities*, p. 89.

Theorem 108. *In order that the inequality*

$$\Phi(a_1')+\cdots+\Phi(a_n')\leqq\Phi(a_1)+\cdots+\Phi(a_n)$$

should be true for all continuous convex functions Φ *it is necessary and sufficient either that* (i) $(a')\prec(a)$, *i.e. that* (a') *is majorised by* (a), *or that* (ii) (a') *is an average of* (a).

If these conditions are satisfied, and $\Phi''(x)$ *exists for all* x *and is positive, then equality can occur if and only if the two sets* (a) *and* (a') *are identical.*

Without loss of generality it can be supposed that

$$a_1'\geqq\cdots\geqq a_n'; \quad a_1\geqq\cdots\geqq a_n.$$

(a') is majorised by (a) means that (*Inequalities*, p. 45)

$$a_1'+\cdots+a_n'=a_1+\cdots+a_n,$$
$$a_1'+\cdots+a_\nu'\leqq a_1+\cdots+a_\nu, \qquad (\nu=1, \ldots, n-1).$$

(a') is an *average* of (a) means that (*Inequalities*, p. 49) there exist n^2 numbers $p_{\mu\nu}$ such that

$$p_{\mu\nu}\geqq 0, \quad \sum_{\mu=1}^{n} p_{\mu\nu}=1, \quad \sum_{\nu=1}^{n} p_{\mu\nu}=1$$

and

$$a_\mu'=p_{\mu 1}a_1+\cdots+p_{\mu n}a_n.$$

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The two statements, „(a') is majorised by (a)", „(a') is an average of (a)" are equivalent. (*Inequalities*, Theorem 46, p. 49).

To obtain inequalities relating to triangles it is enough to take $n=3$. Thus we obtain

Theorem 1. *If ABC, $A'B'C'$ are two triangles whose angles $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are such that $(\alpha)>(\alpha')$ (or what is here the same thing α', β', γ' lie between the least and greatest of the angles α, β, γ) then for any continuous convex Φ*

$$\Phi(\alpha')+\Phi(\beta')+\Phi(\gamma')\leqq\Phi(\alpha)+\Phi(\beta)+\Phi(\gamma),$$

with equality (when $\Phi''(x)>0$) if and only if $\alpha'=\alpha$, $\beta'=\beta$, $\gamma'=\gamma$.

Angles can be replaced by sides if we assume triangles have equal perimeters. Similar statements hold for pairs of spherical triangles.

As a very special case take $\Phi(x)=-\sin kx$. We obtain

$$\sin k\alpha+\sin k\beta+\sin k\gamma\leqq\sin k\alpha'+\sin k\beta'+\sin k\gamma' \tag{1}$$

$(0<k\leqq 2)$ provided $k\alpha\leqq\pi$. Equality occurs if and only if $\alpha=\alpha'$, $\beta=\beta'$, $\gamma=\gamma'$.

Since (i) the angles of an equilateral triangle are majorised by those of any triangle, (ii) the angles of an acute angled triangle are majorised by the acute angled $\left(\frac{1}{2}\pi-\varepsilon, \frac{1}{2}\pi-\varepsilon, 2\varepsilon\right)$ for sufficiently small ε, (iii) the obtuse-angled $\left(\frac{1}{2}\pi+2\varepsilon, \frac{1}{4}\pi-\varepsilon, \frac{1}{4}\pi-\varepsilon\right)$ is majorised for sufficiently small ε by any given obtuse-angled $\left(\alpha>\frac{1}{2}\pi, \beta, \gamma\right)$ which in turn is majorised by the obtuse-angled $(\pi-2\varepsilon, \varepsilon, \varepsilon)$ for sufficiently small ε, it is clear that Theorem 1 leads to inequalities for families of triangles.

Theorem 2. *For any continuous convex Φ,*

$$3\Phi\left(\frac{\pi}{3}\right)\leqq\Phi(\alpha)+\Phi(\beta)+\Phi(\gamma)<\Phi(\pi)+2\Phi(0)$$

(all triangles: equality only for equilateral triangles);

$$3\Phi\left(\frac{\pi}{3}\right)\leqq\Sigma\Phi(\alpha)<2\Phi\left(\frac{1}{2}\pi\right)+\Phi(0)$$

(all acute angled triangles);

$$\Phi\left(\frac{1}{2}\pi\right)+2\Phi\left(\frac{1}{4}\pi\right)<\Sigma\Phi(\alpha)<\Phi(\pi)+2\Phi(0)$$

(all obtuse angled triangles).

Several results in *Geometrical Inequalities* (Groningen, 1969) by Bottema, Djordjević, Janić, Mitrinović, Vasić are special cases of this theorem. For example $\Phi(x)=-\sin x$ ($k=1$ in (1) above) yields **2.2** of **G.I.** As another example suppose that $0<k<1$: then

$$\begin{aligned} 2\sin\frac{1}{2}k\pi<\Sigma\sin k\alpha\leqq 3\sin\frac{1}{3}k\pi, &\qquad \text{(acute triangles)} \\ \sin k\pi<\Sigma\sin k\alpha<\sin\frac{1}{2}k\pi+2\sin\frac{1}{4}k\pi, &\qquad \text{(obtuse triangles).} \end{aligned} \tag{2}$$

For $k=\frac{1}{2}$ compare **G.I. 2.9,**

$$1<\Sigma \sin \frac{1}{2} \alpha \leqq \frac{3}{2} \qquad \text{(all triangles).}$$

It may be noted that the inequality in (2) for acute angled triangle holds for $0<k\leqq 2$.

Further inequalities may be obtained by integration with respect to k. Thus we arrive at

(3) $$\Sigma \frac{\sin^2 k\alpha/2}{\alpha} \leqq \Sigma \frac{\sin^2 k\alpha'/2}{\alpha'}$$

valid whenever (α') is majorised by (α) with equality if and only if $\alpha'=\alpha$, $\beta'=\beta$, $\gamma'=\gamma$. It is assumed that for acute angled trinagles $0<k\leqq 2$: for obtuse angled triangles $0<k\leqq 1$.

In particular then we have the curious inequalities

(4) $$\frac{4}{\pi} \sin^2 \frac{1}{4} k\pi < \Sigma \frac{\sin^2 k\alpha/2}{\alpha} \leqq \frac{9}{\pi} \sin^2 \frac{1}{6} k\pi$$

for all acute triangles $(0<k\leqq 2)$:

(5) $$\frac{1}{\pi} \sin^2 \frac{1}{2} k\pi < \Sigma \frac{\sin^2 k\alpha/2}{\alpha} < \frac{2}{\pi} \sin^2 \frac{1}{4} k\pi + \frac{8}{\pi} \sin^2 \frac{1}{8} k\pi$$

for all obtuse triangles $(0<k\leqq 1)$.

From (4) as a special case $(k=2)$

(6) $$16 R^2 < \pi \Sigma \frac{a^2}{\alpha} \leqq 27 R^2$$

for any acute triangle, equality only for equilateral triangles.

As an example of Theorem 1 applied to products we have

(7) $$\prod \cos k\alpha \leqq \prod \cos k\alpha'$$

whenever $(\alpha')<(\alpha)$, equality only for $\alpha'=\alpha$ etc. $\left(\text{valid for } k>0 \text{ and } k\alpha \leqq \frac{1}{2}\pi\text{:}\right.$ thus for all acute T if $k\leq 1$ and all obtuse T if $\left.k\leqq\frac{1}{2}\right)$. Hence for all acute T

(8) $$\cos^2 \frac{1}{2} k\pi < \prod \cos k\alpha \leqq \cos^3 \frac{1}{3} k\pi \qquad (0<k\leqq 1),$$

equality only for equilateral triangles; and for all obtuse T

(9) $$\cos k\pi < \prod \cos k\alpha < \cos \frac{1}{2} k\pi \, \cos^2 \frac{1}{4} k\pi \qquad \left(0<k\leqq\frac{1}{2}\right).$$

For $k=\frac{1}{2}$ we obtain on the right $(1+\sqrt{2})/4$: it appears that the third inequality in **G.I. 2.28** has $\frac{3}{8}\sqrt{3}$ in place of $(1+\sqrt{2})/4$.

Here is another example:

$$\Sigma \tan k\alpha' \leqq \Sigma \tan k\alpha \tag{10}$$

if $(\alpha')<(\alpha)$, $k\alpha \leqq \frac{1}{2}\pi$ whence

$$\Sigma \tan k\alpha \geqq 3 \tan k\pi/3 \qquad (0<k\leqq 1) \tag{11}$$

all acute T; $\left(\text{For } k=1 \text{ cf. } \textbf{G.I. 2.30}; \ k=\frac{1}{2}, \textbf{ G.I. 2.33.}\right)$: for all obtuse T

$$\tan \frac{1}{2} k\pi + 2 \tan \frac{1}{4} k\pi < \Sigma \tan k\alpha < \tan k\pi \qquad \left(0<k\leqq \frac{1}{2}\right). \tag{12}$$

It is clear that many of these inequalities apply also to pairs of spherical triangles in which the angles (sides) of one are majorised by the angles (sides) of the other.

3. Inequalities for two triangles ABC, $A'B'C'$ when the sides of $A'B'C'$ are majorised by the sides of ABC: specifically $a\geqq b\geqq c$, $a'\geqq b'\geqq c'$, $a+b+c=a'+b'+c'$; a', b', c' lie between the least and greatest of a, b, c.

Theorem 3. *If the sides of $A'B'C'$ are majorised by the sides of ABC, then $F'\geqq F$ and $r'\geqq r$. Equality occurs if and only if the triangles are congruent.*

If T' is not obtuse, then $R'\leqq R$ with equality if and only if the triangles are congruent.

It is only necessary to consider the last part since the inequalities $F'\geqq F$, $r'\geqq r$ follow at once from Theorem 108.

We show that the assumption $R'>R$ leads to a contradiction.

Note that

$$R\sin\gamma \leqq R'\sin\gamma', \quad R'\sin\alpha' \leqq R\sin\alpha,$$

$$R\,\Sigma \sin\alpha = R'\,\Sigma\sin\alpha'.$$

Hence $\sin\alpha'<\sin\alpha$, $\alpha'<\alpha$, since $\alpha'\leqq\frac{1}{2}\pi$ whether $\alpha\geqq\frac{1}{2}\pi$ or not. If also $\gamma\leqq\gamma'$, then $(\alpha')<(\alpha)$ and so by (1) above $\Sigma\sin\alpha<\Sigma\sin\alpha'$ (equality excluded), $R>R'$ contrary to the assumption $R<R'$. Thus $\gamma>\gamma'$. Hence $\beta<\beta'\leqq\pi/2$. But plainly

$$\sin\gamma(\sin\alpha'+\sin\beta'+\sin\gamma')\leqq\sin\gamma'(\sin\alpha+\sin\beta+\sin\gamma)$$

whence

$$\sin\frac{1}{2}\gamma\cos\frac{1}{2}(\alpha'-\beta')\leqq\sin\frac{1}{2}\gamma'\cos\frac{1}{2}(\alpha-\beta),$$

a contradiction since

$$0<\gamma'<\gamma\leqq\frac{1}{2}\pi, \quad 0<\frac{1}{2}(\alpha'-\beta')<\frac{1}{2}(\alpha-\beta)<\frac{1}{2}\pi.$$

It follows that if T' is not obtuse-angled and the sides of T' are majorised by the sides of T, then $R\geqq R'$ and equality occurs if and only the triangles are congruent.

If however T' is obtuse it may happen that $R<R'$.

4. I give now inequalities relating to two triangles T, T' such that $a \leqq a'$, $b \leqq b'$, $c \leqq c'$.

Theorem 4. (i) *If* $a \leqq a'$, $b \leqq b'$, $c \leqq c'$ *and* T' *is not obtuse angled, then* $F \leqq F'$. *If* T' *is obtuse, the inequality need not be true.*

(ii) *If* $a \leqq a'$, $b \leqq b'$, $c \leqq c'$ *and* T *is not obtuse, then* $R \leqq R'$. *If* T *is obtuse, the inequality need not be true.*

(iii) *No necessary relation holds between the in-radii. It is possible to have two acute angled triangles* T, T' *such that*

$$a < a', \quad b < b', \quad c < c' \text{ but } r > r'.$$

Proof. (i) Since $\alpha + \beta + \gamma = \alpha' + \beta' + \gamma'$ it follows that for at least one pair (α, α'), (β, β') (γ, γ'), we have say $\alpha \leqq \alpha'$: $\sin \alpha \leqq \sin \alpha'$ since $\alpha' \leqq \pi/2$. Thus

$$2F' = b'c' \sin \alpha' \geqq bc \sin \alpha = 2F.$$

Plainly the inequality need not be true if T' is obtuse.

(ii) For at least one pair, say (β, β'), $\beta \geqq \beta'$ and now

$$2R \sin \beta' \leqq 2R \sin \beta = b \leqq b' = 2R' \sin \beta', \quad R \leqq R',$$

since $\beta \leqq \pi/2$. Plainly the inequality need not hold if T is obtuse.

(iii) It is sufficient to give a numerical case:

$$a', b', c' = 21, 15, 15: \ a, b, c = 20, 15, 15$$

for which $r' < r$. Choose $k > 1$ so that $kr' < r$. Then the two triangles of sides $(21k, 15k, 15k)$, $(20, 15, 15)$ are both acute angled: the sides of the first exceed the sides of the second but the first in-radius is less than the second in-radius.

5. Theorem 5. *Suppose that a triangle* T *is formed from two triangles* T_1 *and* T_2 *by taking*

$$a = \max(a_1, a_2), \quad b = \max(b_1, b_2), \quad c = \max(c_1, c_2).$$

Then

$$F \geqq \min(F_1, F_2).$$

If one of T_1, T_2 *is non-obtuse, then* $R \geqq \min(R_1, R_2)$.

If both T_1, T_2 *are non-obtuse, then*

$$F \geqq \max(F_1, F_2), \quad R \geqq \max(R_1, R_2).$$

If both T_1 *and* T_2 *are obtuse, no conclusion can be drawn about* R.

Proof. Essentially only two cases arise:

I $\quad a_1 \geqq a_2, \quad b_1 \geqq b_2, \quad c_1 \geqq c_2;$

II $\quad a_1 \leqq a_2, \quad b_1 \geqq b_2, \quad c_1 \geqq c_2.$

In case I, $F = F_1 \geqq \min(F_1, F_2)$: if T_2 is not obtuse, then $F_1 \geqq F_2$. In case II use the relations

$$b_1^2 + c_1^2 - 2b_1 c_1 \cos \alpha_1 = a_1^2 \leqq a_2^2 = b_1^2 + c_1^2 - 2b_1 c_1 \cos \alpha,$$
$$b_2^2 + c_2^2 - 2b_2 c_2 \cos \alpha_2 = a_2^2 \qquad = b_1^2 + c_1^2 - 2b_1 c_1 \cos \alpha.$$

If $\alpha \leqq \frac{1}{2}\pi$, then $0 \leqq \cos\alpha \leqq \cos\alpha_1$ so that $\alpha_1 \leqq \alpha$. Thus

$$2F = b_1c_1 \sin\alpha \geqq b_1c_1 \sin\alpha_1 = 2F_1.$$

If however $\alpha > \frac{1}{2}\pi$, then

$$2b_2c_2 \cos(\pi - \alpha_2) \geqq 2b_1c_1 \cos(\pi - \alpha) > 0,$$

$$0 < \pi - \alpha_2 \leqq \pi - \alpha < \frac{1}{2}\pi, \quad \sin\alpha_2 \leqq \sin\alpha,$$

$$2F = b_1c_1 \sin\alpha \geqq b_2c_2 \sin\alpha_2 = 2F_2.$$

In any case $F \geqq \min(F_1, F_2)$.

The rest of Theorem 5 is simply an application of Theorem 4. To see that no conclusion can be drawn about R if both T_1 and T_2 are obtuse take T_1 to have sides $2, 1+\varepsilon, 1+\varepsilon$ and T_2 to have sides $1+\varepsilon, 2, 1+\varepsilon$ for small enough positive ε.

6. I conclude with an inequality relating to the areas of three triangles whose sides are connected by the equations

(1) $$a = (a_1^p + a_2^p)^{1/p}, \quad b = (b_1^p + b_2^p)^{1/p}, \quad c = (c_1^p + c_2^p)^{1/p} \qquad (p \geqq 1).$$

Theorem 6. *The areas of triangles T, T_1, T_2 whose sides satisfy* (1) *are such that*

(2) $$F^{p/2} \geqq F_1^{p/2} + F_2^{p/2} \qquad (p = 1, 2, 4).$$

Equality occurs if and only if T_1, T_2 are similar.

For any $p > 4$ (integer or not) there exist triangles T_1, T_2 and T for which the inequality is false.

REMARKS. 1. I gave the case $p = 2$ as a problem some years ago: see **G.I. 10.12.** The inequality is equivalent to PEDOE's two-triangle inequality **(G.I. 10.8)**

$$\Sigma a_1^2(b_2^2 + c_2^2 - a_2^2) \geqq 16F_1F_2;$$

equality if and only if T_1 and T_2 are similar: it is also equivalent to the inequality

$$(\lambda a^2 + \mu b^2 + \nu c^2)^2 \geqq 16(\mu\nu + \nu\lambda + \mu\lambda)F^2$$

for any triangle T of sides a, b, c and area F: the last inequality is essentially **G.I. 14.1.**

2. It would be interesting to determine whether the inequality (2) holds $1 \leqq p \leqq 4$.

***Proof for* $p = 1$.** Plainly

$$s = s_1 + s_2, \quad s - a = (s_1 - a_1) + (s_2 - a_2), \text{ etc.}$$

where s, s_1, s_2 are the respective semi-perimeters. MINKOWSKI's inequality applied to $F^{\frac{1}{2}} = \{s(s-a)(s-b)(s-c)\}^{\frac{1}{4}}$ yields

$$F^{\frac{1}{2}} \geqq F_1^{\frac{1}{2}} + F_2^{\frac{1}{2}},$$

equality only for similar triangles.

In the same way for any integer $n \geqq 5$

(3) $$2^{(n-4)/n} F^{2/n} \geqq F_1^{2/n} + F_2^{2/n},$$

equality only for congruent triangles T_1, T_2.

Proof for $p=2$. I give my original proof which depends on a classical inequality of MINKOWSKI for definite real matrices. Suppose that $A=(a_{ik})$, $B=(b_{ik})$ are positive definite $n \times n$ matrices: then $C=A+B$ is positive definite and the determinants satisfy the inequality

(4) $$|c_{ik}|^{1/n} \geqq |a_{ik}|^{1/n} + |b_{ik}|^{1/n},$$

equality if and only if A and B are proportional. (See e.g. **HLP**, *Inequalities*, p. 35.)

Take

$$A = \begin{Vmatrix} b_1^2 & b_1 c_1 \cos \alpha_1 \\ b_1 c_1 \cos \alpha_1 & c_1^2 \end{Vmatrix} \qquad B = \begin{Vmatrix} b_2^2 & b_2 c_2 \cos \alpha_2 \\ b_2 c_2 \cos \alpha_2 & c_2^2 \end{Vmatrix}$$

so that

$$C = \begin{Vmatrix} b^2 & bc \cos \alpha \\ bc \cos \alpha & c^2 \end{Vmatrix}$$

and the inequality $F \geqq F_1 + F_2$ is simply the case $n=2$ of MINKOWSKI's inequality.

It is also the case that

(5) $$\frac{|C|}{|C_1|} \geqq \frac{|A|}{|A_1|} + \frac{|B|}{|B_1|}$$

(where A_1, B_1, C_1 are corresponding principal minors). This inequality which I gave in 1930 (*Journal London Math. Soc.* **5** (1930), 114-119), is known as BERGSTROM's Inequality after BERGSTROM who discovered it independently (and realised its importance) in 1949 (see reference p. 90 in BECKENBACH and BELLMAN, *Inequalities*, Berlin 1961.) As an application (5) yields for $n=2$ and the matrices A, B the inequality

(6) $$h^2 \geqq h_1^2 + h_2^2$$

for corresponding altitudes of T, T_1, T_2.

Plainly corresponding inequalities hold in E^n: for example if tetrahedra T, T_1, T_2 have edges connected by the equations

$$a^2 = a_1^2 + a_2^2, \ldots, \ldots$$

then their volumes satisfy the inequalities

(7) $$V^{2/3} \geqq V_1^{2/3} + V_2^{2/3},$$

equality if and only if T_1 and T_2 are similar: their corresponding altitudes satisfy the inequalities

$$h^2 \geqq h_1^2 + h_2^2.$$

Proof for $p=4$. It is enough to note that

$$8(F^2 - F_1^2 - F_2^2) = \Sigma \{(b_1^4 + b_2^4)^{\frac{1}{2}} (c_1^4 + c_2^4)^{\frac{1}{2}} - (b_1^2 c_1^2 + b_2^2 c_2^2)\}$$

and

$$(b_1^4+b_2^4)(c_1^4+c_2^4)=(b_1^2c_1^2+b_2^2c_2^2)^2+(b_1^2c_2^2-b_2^2c_1^2)^2$$

so that $F^2 \geqq F_1^2+F_2^2$ with equality only for similar T_1, T_2.

To show that the inequality (2) cannot hold for $p>4$ and all triangles T_1, T_2 it is sufficient to take a degenerate case; T_1 of sides $2x$, x, x (x large) and T_2 of sides 1, 1, 1. Then for large x an easy calculation gives

$$F^2 \approx 2(1-2^{-p})/px^{p-4}, \quad F_1=0, \quad F_2^2=3/16$$

so that the inequality $F^{p/2} \geqq F_1^{p/2}+F_2^{p/2}$ cannot hold for large x if $p>4$.

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