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361. A NOTE ON SYMMETRIC FORMS*

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Let $p_1, \dots, p_m$ denote m-positive real numbers and p denote the smallest of these m-positive real numbers. If $a_1, \dots, a_m$ are non-negative real numbers and

$$(1.1)\qquad H_r(p_1, \dots, p_m) = \sum \binom{p_1+i_1-1}{i_1} \cdots \binom{p_m+i_m-1}{i_m} a_1^{i_1} \cdots a_m^{i_m}$$

$$(i_1 + \cdots + i_m = r),$$

$$(1.2)\qquad E_r(p_1, \dots, p_m) = \sum \binom{p_1}{i_1} \cdots \binom{p_m}{i_m} a_1^{i_1} \cdots a_m^{i_m} \qquad (i_1 + \cdots + i_m = r),$$

then in [1], it is proved that

$$(1.3)\qquad \{H_r(p_1, \dots, p_m)\}^{\frac{1}{r}} \geqq \{H_{r+1}(p_1, \dots, p_m)\}^{\frac{1}{r+1}}$$

where $p_i \geqq 1$ for all i, and

$$(1.4)\qquad \{E_r(p_1, \dots, p_m)\}^{\frac{1}{r}} \geqq \{E_{r+1}(p_1, \dots, p_m)\}^{\frac{1}{r+1}}$$

where $r < p+1$, p not integral.

In this paper we prove that the inequalities (1.3) and (1.4) are satisfied by a more general type of non-symmetric functions which are defined in the following way.

$$(1.5)\qquad H_r(q;\, p_1, \dots, p_m) = \sum \begin{bmatrix} p_1+i_1-1 \\ i_1 \end{bmatrix} \cdots \begin{bmatrix} p_m+i_m-1 \\ i_m \end{bmatrix} a_1^{i_1} \cdots a_m^{i_m}$$

$$(i_1 + \cdots + i_m = r),$$

$$(1.6)\qquad E_r(q;\, p_1, \dots, p_m) = \sum \begin{bmatrix} p_1 \\ i_1 \end{bmatrix} \cdots \begin{bmatrix} p_m \\ i_m \end{bmatrix} a_1^{i_1} \cdots a_m^{i_m} \qquad (i_1 + \cdots + i_m = r),$$

where $\begin{bmatrix} p \\ k \end{bmatrix}$ denote the q-binomial coefficient.

The q-binomial coefficients are defined by

$$\begin{bmatrix} p \\ k \end{bmatrix} = \frac{(1-q^p)\cdots(1-q^{p-k+1})}{(1-q)\cdots(1-q^k)}, \qquad \begin{bmatrix} p \\ k \end{bmatrix} = 0 \quad (k<0) \quad \text{and} \quad \begin{bmatrix} p \\ 0 \end{bmatrix} = 1.$$

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We note that when q tends to one, then

(1.7) $$H_r(1;\, p_1, \dots, p_m) = H_r(p_1, \dots, p_m),$$

and

(1.8) $$E_r(1;\, p_1, \dots, p_m) = E_r(p_1, \dots, p_m)$$

also $H_r(0;\, p_1, \dots, p_m)$ is the complete symmetric function and $E_r(0;\, p_1, \dots, p_m)$ is the elementary symmetric function. In [1], the author proved inequalities (1.3) and (1.4) by making use of EULER's theorem on homogeneous functions. The same method cannot be applied here.

Definition. *A sequence $\{a_r\}$ is called a logarithmically concave sequence if*

$$a_r^2 \geqq a_{r-1} a_{r+1}.$$

For a logarithmically concave sequence the following theorem is provod in [2]

Theorem A. *If $\{p_r\}$ and $\{q_r\}$ are two positive logarithmically concave sequences. with $p_0 = q_0 = 1$, then the sequence $\{W_r\}$ is also logarithmically concave where W_r is given by the formal power series*

$$\left(\sum p_r x^r\right)\left(\sum q_r x^r\right) = \left(\sum w_r x^r\right).$$

Theorem 1.

$$\{H_r(q;\, p_1, \dots, p_m\}^{\frac{1}{r}} \geqq \{H_{r+1}(q;\, p_1, \dots, p_m)\}^{\frac{1}{r+1}}$$

where $p_i \geqq 1$, for all i and $(0 \leqq q < +\infty)$. The inequality is strict unless $p_1 = \dots = p_m = 1$, $q \to 1$, and $m = 1$.

Proof. We first prove that the sequence

(2.1) $$\left\{\begin{bmatrix} p_t + i_t - 1 \\ i_t \end{bmatrix} a_t^{i_t}\right\}$$

is a positive logarithmically concave sequence for all $t = 1, \dots, m$.

To prove this it is enough if we prove that

(2.2) $$\begin{bmatrix} p_t + i_t - 1 \\ i_t \end{bmatrix}^2 - \begin{bmatrix} p_t + i_t - 2 \\ i_t \end{bmatrix}\begin{bmatrix} p_t + i_t \\ i_t \end{bmatrix} \geqq 0.$$

Now

$$\begin{bmatrix} p_t + i_t - 1 \\ i_t \end{bmatrix}^2 - \begin{bmatrix} p_t + i_t - 2 \\ i_t \end{bmatrix}\begin{bmatrix} p_t + i_t \\ i_t \end{bmatrix} = \begin{bmatrix} p_t + i_t - 2 \\ i_t - 1 \end{bmatrix}^2 \left(\frac{1 - q^{p_t + i_t - 1}}{1 - q^{i_t}}\right)\left\{\frac{1 - q^{p_t + i_t - 1}}{1 - q^{i_t}} - \frac{1 - q^{p_t + i_t}}{1 - q^{i_t + 1}}\right\}.$$

Hence

$$\begin{bmatrix} p_t + i_t - 1 \\ i_t \end{bmatrix}^2 - \begin{bmatrix} p_t + i_t - 2 \\ i_t \end{bmatrix}\begin{bmatrix} p_t + i_t \\ i_t \end{bmatrix} =$$

$$\begin{bmatrix} p_t + i_t - 2 \\ i_t \end{bmatrix}^2 \frac{1 - q^{p_t + i_t - 1}}{(1 - q^{i_t})^2 (1 - q^{i_t + 1})} \{(1 - q)(q^{i_t} - q^{p_t + i_t - 1})\}.$$

Since $p_t \geqq 1$, we find that (2.2) is true for all $t = 1, \dots, m$. Now applying theorem A we find that

(2.3) $$\{H_r(q;\, p_1, \dots, p_m)\}^2 \geqq H_{r-1}(q;\, p_1, \dots, p_m)\, H_{r+1}(q;\, p_1, \dots, p_m).$$

Now the theorem can be deduced from (2.2) as in [3].

Theorem 2.

$$\{E_r(q;\ p_1, \ldots, p_m)\}^{\frac{1}{r}} \geqq \{E_{r+1}(q;\ p_1, \ldots, p_m)\}^{\frac{1}{r+1}}$$

where $p+1>r$ *and* $0 \leqq q < +\infty$. *The inequality is strict unless* $p_1 = \cdots = p_m = 1$ $q \to 1$ *and* $m=1$.

Proof. Same as theorem 1.

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