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360. SOME INEQUALITIES FOR THE TRIANGLE*

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1. Let ABC be an arbitrary triangle and let P denote a point in the interior of ABC. Let D, E, F denote the feet of the perpendiculars from P to BC, CA, AB, respectively. Let a, b, c denote the sides and α, β, γ the angles of ABC. Following the notation of [1], put

$$R_1=PA, \qquad R_2=PB, \qquad R_3=PC,$$
$$r_1=PD, \qquad r_2=PE, \qquad r_3=PF.$$

Let K denote the area of ABC and K_0 the area of DEF. We shall show first that

$$K_0 \leqq \frac{1}{4} K, \tag{1.1}$$

with equality if and only if P is the circumcenter of ABC.

***Proof of* (1.1).** Clearly

$$2K_0=ar_1+br_2+cr_3. \tag{1.2}$$

On the other hand, since

$$\sphericalangle EPF=180^\circ-\alpha,$$

it follows that

$$\text{area } EPF=\frac{1}{2} r_2 r_3 \sin \alpha.$$

Similarly

$$\text{area } FPD=\frac{1}{2} r_3 r_1 \sin \beta, \qquad \text{area } DPE=r_1 r_2 \sin \gamma,$$

so that

$$2K_0=r_2 r_3 \sin \alpha+r_3 r_1 \sin \beta+r_1 r_2 \sin \gamma. \tag{1.3}$$

Since

$$a=2R\sin \alpha, \qquad b=2R\sin \beta, \qquad c=2R\sin \gamma,$$

where R is the circumradius of ABC, (1.3) is equivalent to

$$4K_0R=ar_2 r_3+br_3 r_1+cr_1 r_2. \tag{1.4}$$

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It follows that

$$\begin{aligned}4\,cK_0R &= c^2 r_1 r_2 + (ar_2 + br_1)\,cr_3\\ &= c^2 r_1 r_2 + (ar_2 + br_1)\,(2\,K - ar_1 - br_2)\\ &= 2\,(ar_2 + br_1)\,K - [abr_1^2 + abr_2^2 + (a^2 + b^2 - c^2)\,r_1 r_2]\\ &= 2\,(ar_2 + br_1)\,K - ab\,(r_1^2 + r_2^2 + 2\,r_1 r_2 \cos\gamma)\\ &= 2\,(ar_2 + br_1)\,K - ab\,[(r_1 + r_2\cos\gamma)^2 - r_2^2 \sin^2\gamma]\\ &= ab\,(\lambda^2 + \mu^2) - ab\,(r_1 + r_2\cos\gamma - \lambda)^2 - ab\,(r_2 \sin\gamma - \mu)^2,\end{aligned}$$

where

$$ab\,\lambda = bK,$$

$$ab\,\lambda\cos\gamma + ab\,\mu\sin\gamma = aK.$$

It is easily verified that

$$\lambda = \frac{K}{a} = \frac{bc}{4R} = R\sin\beta\sin\gamma,$$

$$\begin{aligned}\mu &= \frac{K}{ab\sin\gamma}\,(a - b\cos\gamma) = \frac{c}{4\,R\sin\gamma}\,(a - b\cos\gamma)\\ &= \frac{1}{2}\,(a - b\cos\gamma) = \frac{1}{2}\,c\cos\gamma = R\cos\beta\sin\gamma,\end{aligned}$$

$$\lambda^2 + \mu^2 = R^2 \sin^2\gamma = \frac{1}{4}\,c^2.$$

Therefore

(1.5) $$4\,cK_0R \leqq \frac{1}{4}\,abc^2,$$

which evidently proves (1.1). Moreover we have equality if and only if

$$\begin{cases} r_1 + r_2\cos\gamma = R\sin\beta\sin\gamma,\\ \quad r_2\sin\gamma = R\cos\beta\sin\gamma.\end{cases}$$

This implies

$$r_1 = R\cos\alpha, \qquad r_2 = R\cos\beta, \qquad r_3 = R\cos\gamma,$$

so that P is the circumcenter of ABC.

2. In the next place consider

$$\begin{aligned}&(ar_1 + br_2 + cr_3)\left(\frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3}\right)\\ &\quad = a^2 + b^2 + c^2 + bc\left(\frac{r_2}{r_3} + \frac{r_3}{r_1}\right) + ca\left(\frac{r_3}{r_1} + \frac{r_1}{r_3}\right) + ab\left(\frac{r_1}{r_2} + \frac{r_2}{r_1}\right)\\ &\quad \geqq a^2 + b^2 + c^2 + 2\,bc + 2\,ca + 2\,ab\\ &\quad = (a + b + c)^2,\end{aligned}$$

with equality if and only if $r_1 = r_2 = r_3$. Thus

(2.1) $$(ar_1 + br_2 + cr_3)\,(ar_2 r_3 + br_3 r_1 + cr_1 r_2) \geqq (a + b + c)^2\, r_1 r_2 r_3,$$

with equality if and only if P is the incenter of ABC. Hence by (1.2) and (1.4) we have

(2.2) $$2KK_0R \geqq s^2 r_1 r_2 r_3,$$

with equality if and only if P is the incenter of ABC.

The sides of DEF are given by

(2.3) $$EF = R_1 \sin \alpha, \qquad FD = R_2 \sin \beta, \qquad DE = R_3 \sin \gamma.$$

Applying the relation

(2.4) $$abc = 4KR$$

to the triangle DEF we get

$$R_1R_2R_3 \sin \alpha \sin \beta \sin \gamma = 4K_0R_0,$$

where R_0 is the circumradius of DEF. It follows that

(2.5) $$abc\, R_1R_2R_3 = 32 K_0R_0R^3.$$

Applying (2.4) again, (2.5) becomes

(2.6) $$R_1R_2R_3K = 8 K_0R_0R^2.$$

If we now multiply (2.6) by K and make use of (2.2), we get

(2.7) $$R_1R_2R_3K^2 \geqq 4 s^2 R_0Rr_1 r_2 r_3.$$

Since $K = rs$, where r is the radius of the incircle of ABC, (2.7) reduces to

(2.8) $$R_1R_2R_3r^2 \geqq 4 R_0Rr_1 r_2 r_3$$

with equality if and only if P is the incenter of ABC.

If we multiply (2.6) by $2K_0R$ and then use (2.2), we get

$$16 K_0^2R_0R^3 = 2 r_1 r_2 r_3 KK_0R \geqq R_1R_2R_3 r_1 r_2 r_3 s^2,$$

so that

(2.9) $$16 K_0^2R_0R^3 \geqq R_1R_2R_3 r_1 r_2 r_3 s^2$$

with equality if and only if P is the incenter of ABC. Since [**1**, 12.26]

(2.10) $$R_1R_2R_3 \geqq 8 r_1 r_2 r_3,$$

with equality if and only if ABC is equilateral and P is the incenter, (2.9) yields

(2.11) $$2 K_0^2R_0R^3 \geqq (r_1 r_2 r_3 s)^2$$

with equality if and only if ABC is equilateral and P is the incenter.

If we rewrite (1.1) in the form

$$\frac{4 R_1R_2R_3 \sin \alpha \sin \beta \sin \gamma}{R_0} \leqq \frac{abc}{R},$$

it is evident that we get the inequality

(2.12) $$R_1R_2R_3 \leqq 2 R_0R^2$$

with equality if and only if P is the circumcenter of ABC.

Combining (2.12) with (2.8), we get

$$4\,r_1\,r_2\,r_3R_0R \leqq 2\,r^2R_0R^2,$$

so that

(2.13) $$2\,r_1\,r_2\,r_3 \leqq r^2R$$

with equality if and only if ABC is equilateral and P is the incenter.

Again, by (2.9) and (1.1),

$$R_1R_2R_3r_1\,r_2\,r_3\,s^2 \leqq K^2R_0R^3,$$

which is equivalent to

(2.14) $$R_1R_2R_3r_1\,r_2\,r_3 \leqq r^2\,R_0R^3$$

with equality if and only if ABC is equilateral and P is the incenter.

Returning to (1.2), since

$$2\,K = ar_1 + br_2 + cr_3 \geqq 3\,(abcr_1\,r_2\,r_3)^{\frac{1}{3}},$$

we get

$$8\,K^3 \geqq 27\,abcr_1\,r_2\,r_3.$$

Therefore [**1**, 12.29]

(2.15) $$27\,Rr_1\,r_2\,r_3 \leqq 2\,K^2$$

with equality if and only if P is the centroid of ABC.

Multiplying (2.13) and (2.15) we get

$$27\,(r_1\,r_2\,r_3)^2 \leqq r^2\,K^2,$$

so that

(2.16) $$3\sqrt{3}\,r_1\,r_2\,r_3 \leqq rK = r^2\,s,$$

with equality if and only if ABC is equilateral and P is the incenter.

We remark that (2.15) is stronger than (2.13). This is a consequence of the inequality [**1**, 5.3]

(2.17) $$27\,R^2 \geqq 4\,s^2.$$

By (2.15) and (2.17)

$$2\,K^2R \geqq 27\,r_1\,r_2\,r_3\,R^2 \geqq 4\,r_1\,r_2\,r_3\,s^2,$$

which evidently implies (2.13).

REFERENCE

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