PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE: MATHÉMATIQUES ET PHYSIQUE

№ 338 — № 352 (1971)

345. THE SUBPOLAR FORM OF A LINEAR TRANSFORMATION*

A. R. Amir-Moéz, W. A. Donnell, C. R. Perry

For a singular linear transformation A on an n-dimensional unitary space, the unitary part of the polar decomposition is not unique [**1**, p. 169]. In this note we study a decomposition of A called the subpolar decomposition of A in which the factors are unique.

1. Definitions and notations. We shall use the standard notations of linear algebra of a unitary space E_n. For a linear trasformation A on E_n the adjoint A^* is defined by $(A\xi, \eta)=(\xi, A^*\eta)$. The generalized inverse of A will be denoted by A^- [**3**]. It is clear that $AA^-=P$ and $A^-A=Q$, where P is the Hermitian projection on the range of A and Q is the Hermitain projection on the range of A^*.

A linear transormation A on E_n is called subunitary if $A^-=A^*$. One can easily see that a subunitary transformation is the same as a partial isometry [**1**, p. 150].

2. Theorem. *Let A be a subunitary transformation on E_n. Then:*

i) *Each proper value of A has absolute value less than or equal to one;*

ii) *Every non-zero proper value of A has absolute value one if only if A is normal;*

iii) *If ξ is a proper vector corresponding to the non-zero proper value m, then $||A^*\xi||=||\xi||$;*

iv) *Let $m_1 \neq m_2$ be two non-zero proper values of A with ξ_1, ξ_2 the corresponding proper vectors. Then $(\xi_1, \xi_2)=(A^*\xi_1, A^*\xi_2)$.*

The proof will be omitted.

3. Theorem. *A necessary and sufficient condition for a linear transformation A on E_n to be subunitary is that A is a product of a unitary transformation and a Hermitian projection.*

(The proof is omitted.)

---

* Presetend December 25, 1970 by D. S. MITRINOVIĆ.

This theorem enables one to choose an orthonormal basis with respect to which the matrix of A has the form:

$$\begin{pmatrix} u_{11} \cdots u_{1n} \\ u_{k1} \cdots u_{kn} \\ 0 \cdots 0 \\ 0 \cdots 0 \end{pmatrix},$$

where k is the rank of A and the set $\{(n_{i1}, \ldots, u_{in})\}$ is orthonormal. Similarly one can obtain a matrix of the form:

$$\begin{pmatrix} v_{11} \cdots v_{1k} & 0 \cdots 0 \\ \vdots \quad \cdots & \\ v_{n1} \cdots v_{nk} & 0 \cdots 0 \end{pmatrix},$$

where $\{(v_{n1}, \ldots, v_{nj})\}$ is orthonormal.

4. Theorem. *Let A be a linear transformation of rank k on E_n. Then there exist a unique subunitary transformation S with $S^- S = A^- A$, and a unique positive semi-definite transformation K of rank k such that $A = SK$. This form is called the subpolar form of A.*

Proof. It is clear that we may put $S = A^* \sqrt{A^* A}$ and $K = \sqrt{A^* A}$. This proves the existence of S and K.

Now suppose $A = S_1 K_1 = S_2 K_2$. This implies $K_1{}^2 = K_1 S_1^- S_1 K_1 = K_2 S_2^- S_2 K_2 = K_2{}^2$. Thus $K_1 = K_2 = \sqrt{A^* A}$. On the other hand we observe that $S_1 S_1^- = S_2 S_2^- = P$, where P is the Hermitian projection on the range of A. This with $S_1 \sqrt{A^* A} = S_2 \sqrt{A^* A}$ imlies the uniqueness of S.

5. Corollary. *Let A satisfy the hypothesis of* 4. *Then A can be uniquely written as $A = \sqrt{AA^*} S$, where S is the same subunitary transformation of* 4.

Proof. It is clear that $A = S\sqrt{A^* A}$ implies $A^* = \sqrt{A^* A} S^*$. Thus one may conclude that $\sqrt{AA^*} = S\sqrt{A^* A} S^*$ which implies $\sqrt{AA^*} S = S\sqrt{A^* A}$. This proves the corollary.

6. Theorem. *Let $A = SK$ be the subpolar form of A. Then A is normal if and only if $SK = KS$.*

The proof is very simple and will be omitted.

7. Theorem (HESTENES [2]). *Let A be a non-zero linear transformation on E_n. Let $m_1 > \cdots > m_h > 0$ be the distinct absolute singular values of A. Then A has a unique decomposition*

$$A = m_1 S_1 + \cdots + m_h S_h,$$

where S_i, $i = 1, \ldots, h$ are subunitary and $S_1 + \cdots + S_h = S$, the subunitary part of A.

The proof is clear if one considers the diagonal form of $\sqrt{A^* A}$.

REFERENCES

1. P. R HALMOS: *Finite-Dimensional Vector Spaces*, 2nd ed. New York, 1958.
2. M. R. HESTENES: *Relative Hermitian matrices.* Pacific J. Math. **11** (1961), 225—245.
3. R. PENROSE: *A generalized inverse for matrices.* Proc. Cambridge Philos. Soc **51** (1955), 406—413.

Texas Tech University
Lubbock, USA