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341. SOME INEQUALITIES FOR THE MEDIANS OF A TRIANGLE*

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In this paper I shall derive some inequalities concerning the medians of a plane triangle not found in GI[1] and hence thought to be new. In notations we shall follow GI, and all decimal references will be to its sections. In a few instances we shall use the well-known shorthand notation explained in **[1]**, 4.

Most proofs make use of the following observation. Let G be the centroid of triangle ABC and $BGCD$ a parallelogram. Consider the triangle BGD. The reader will readily see that its sides are $\frac{2}{3}m_a, \frac{2}{3}m_b, \frac{2}{3}m_c$, its medians $\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c$ and its area $\frac{1}{3}F$. We conclude that the semiperimeter is

$$\frac{m_a+m_b+m_c}{3},$$

the inradius

$$\frac{F}{m_a+m_b+m_c},$$

and the circumradius

$$\frac{2\,m_a m_b m_c}{9F}.$$

Hence, every inequality containing only quantities mentioned above may be translated into another inequality only by using it on triangle BGD.

First we shall remark that 4.14 and 4.21 of GI together constitute the inequality

$$M_\lambda^2(a, b, c) \geqq \frac{4}{\sqrt{3}}F \qquad (\lambda \geqq 0),$$

(of which 4.10 is the case $\lambda=4$, 4.4 the case $\lambda=2$, and $s \geqq 3\sqrt{3}r$ the case $\lambda=1$). Translating this inequality (as explained above) we get

$$(1) \qquad M_\lambda^2(m_a, m_b, m_c) \geqq F\sqrt{3} \qquad (\lambda \geqq 0).$$

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* Presented October 1, 1970 by O. Bottema and R. R. Janić.

[1] O. Bottema, R. Ž. Đorđević, R. R. Janić, D. S. Mitrinović, P. M. Vasić: *Geometric Inequalities*. Groningen 1969.

Special cases of (1) found in GI are 8.4 ($\lambda=1$) and 8.6 ($\lambda=2$). Translation of 4.10 yields the case $\lambda=4$:

(2) $$m_a^4+m_b^4+m_c^4 \geqq 9F^2.$$

As $m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)$ we get by 5.14:

(3) $$9r(2R-r) \leqq m_a^2+m_b^2+m_c^2 \leqq 3(2R^2+r^2).$$

Translating 4.7 we get (in shorthand notation):

$$\frac{1}{2}\sum m_a^2+\frac{3\sqrt{3}}{2}F \leqq \sum m_b m_c \leqq \frac{5}{6}\sum m_a^2+\frac{\sqrt{3}}{2}F.$$

This chain of inequalities we weaken by using the estimations (3) and (see, for example, 5.3 and 5.11)

$$3\sqrt{3}\,r \leqq s \leqq \frac{3\sqrt{3}}{2}R.$$

In this manner we get:

(4) $$9r(R+r) \leqq m_b m_c+m_c m_a+m_a m_b \leqq 5R^2+\frac{9}{4}Rr+\frac{5}{2}r^2.$$

Using $R \geqq 2r$ (see 5.1, or **[1]**, 11) we weaken (4) thereby getting the more pleasing chain of inequalities:

(5) $$9r(R+r) \leqq m_b m_c+m_c m_a+m_a m_b \leqq (5R-r)(R+r)$$

We multiply (5) by 2, add (3) to the resulting chain and get:

(6) $$9r(4R+r) \leqq (m_a+m_b+m_c)^2 \leqq (4R+r)^2.$$

The right hand part of (6) is equivalent to 8.2 (the proof of which given in GI is not valid in the obtuse case).

Translating $s \leqq \frac{3\sqrt{3}}{2}R$ we get the inequality

$$\frac{1}{3}(m_a+m_b+m_c) \leqq \frac{3\sqrt{3}}{2}\cdot\frac{2m_a m_b m_c}{9F},$$

which we shall restate in the following more appealing form:

(7) $$\frac{1}{m_b m_c}+\frac{1}{m_c m_a}+\frac{1}{m_a m_b} \leqq \frac{\sqrt{3}}{F}.$$

By (7) and the arithmetic-harmonic mean inequality, or by translation of 4.5, we get:

(8) $$m_b m_c+m_c m_a+m_a m_b \geqq 3\sqrt{3}\,F.$$

Whether (8) or the left hand inequality of (5) is the stronger one, depends on the triangle (for details, see **[1]**, 37).

Translating 4.12 we get the inequality

$$m_b^2 m_c^2 + m_c^2 m_a^2 + m_a^2 m_b^2 \geqq 9F^2, \tag{9}$$

which is stronger than (2) because $yz+zx+xy \leqq x^2+y^2+z^2$ for non-negative numbers x, y, z.

We shall now prove the inequality

$$am_a^2 + bm_b^2 + cm_c^2 \geqq 9RF, \tag{10}$$

which we may restate as

$$\frac{m_a^2}{bc} + \frac{m_b^2}{ca} + \frac{m_c^2}{ab} \geqq \frac{9}{4}. \tag{11}$$

Using the shorthand notation we have

$$\sum am_a^2 = \sum a\left(\frac{1}{4}b^2 + \frac{1}{4}c^2 + \frac{1}{2}bc\cos\alpha\right) = \frac{1}{4}\sum(ab^2+a^2b) + \frac{1}{2}abc\sum\cos\alpha.$$

Now $abc = 4RF = 4Rrs$, $\sum \cos\alpha = \frac{R+r}{R}$ (see [1], 10) and

$$\sum(a^2b+ab^2) = (\sum a)(\sum bc) - 3abc$$
$$= 2s(s^2+4Rr+r^2) - 12Rrs = 2s(s^2-2Rr+r^2)$$

(see [1], 9). Hence we get

$$am_a^2 + bm_b^2 + cm_c^2 = \frac{1}{2}s(s^2+2Rr+5r^2).$$

As $s^2+5r^2 \geqq 16Rr$ (see 5.8) and $rs=F$ we have (10).

By translation of (11) we get:

$$\frac{a^2}{m_b m_c} + \frac{b^2}{m_c m_a} + \frac{c^2}{m_a m_b} \geqq 4. \tag{12}$$

The reader may compare 5.30. Dividing (12) by $4R^2$ we get:

$$\frac{\sin^2\alpha}{m_b m_c} + \frac{\sin^2\beta}{m_c m_a} + \frac{\sin^2\gamma}{m_a m_b} \geqq \frac{1}{R^2}. \tag{13}$$

In 5.45 it is proved that $r_a^2+r_b^2+r_c^2 \geqq 8R^2-5r^2$. As a consequence of $R \geqq 2r$ we have $8R^2-5r^2 \geqq 3(2R^2+r^2)$. Hence, comparing with the right hand inequality of (3), we get:

$$m_a^2+m_b^2+m_c^2 \leqq r_a^2+r_b^2+r_c^2. \tag{14}$$

Combining (14) with 8.19 and some trivialities we get:

$$\sum h_a^2 \leqq \sum w_a^2 \leqq s^2 \leqq \sum m_a^2 \leqq \sum r_a^2.$$

The subsections 5.33 (see the proof), 6.14, 8.9 and 8.20 of GI provide the following three chains:

$$\sum h_a \leqq \sum w_a \leqq \sum m_a \leqq \sum r_a,$$

$$\sum w_a \leqq 3(R+r) \leqq \sum r_a,$$

$$\sum w_a \leqq s\sqrt{3} \leqq \sum r_a.$$

In each inequality stated above equality holds if and only if the triangle is equilateral.

REFERENCE

1. A. Bager: *A Family of Goniometric inequalities.* Same Publications pp. 5—25.

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