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340. INEQUALITIES FOR R, r AND s*

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1. Introduction. In this paper R and r stand for the radii of the circumcircle and the inscribed circle of a triangle ABC and s for its semi-perimeter. By various methods a set of inequalities between R, r and s have been derived from EULER's time to recent years. The most important contribution to the subject seems to us a paper of BLUNDON[1], who has drawn attention to an inequality (known for many years) which we should like to call absolute or *fundamental*: it is not only valid for every triangle, but inversely, if it is satisfied by R, r and s there exists a triangle with these data. In other words, it is not only a necessary but also a sufficient condition. This means that it is the best condition available; it can not be improved and, in principle, all other inequalities for R, r and s are consequences of this one. Moreover BLUNDON has given an illustrative geometric interpretation by mapping the classes of similar triangles on the points of a set S in the Euclidean plane; he determines the boundary of S by means of two irrational functions and makes use of the image to prove several inequalities, and in particular the strongest linear inequalities and the second order ones of a certain type.

For the sake of completeness we derive in 2 the fundamental inequality. In 3 we introduce a geometric mapping slightly different from BLUNDON's and we prove that the boundary of S is (a part of) a well-known curve: STEINER's *hypocycloid*, sometimes called *deltoid*. In 4 we verify some known linear inequalities by means of the geometric interpretation. 5 deals with quadratic inequalities; we mention in particular a proof of that given by NAKAJIMA, to which another inequality is added which improves it. In 6 some higher order inequalities are considered; it contains a proof of an inequality of the fourth order conjectured by BAGER.

2. The fundamental inequality. If a, b, c are the sides of the triangle we introduce the expressions $u_1 = -a+b+c$, $u_2 = a-b+c$, $u_3 = a+b-c$. The necessary and sufficient conditions for the existence of the triangle are: $u_i > 0$ ($i = 1, 2, 3$). Indeed, as $2a = u_2 + u_3$, $2b = u_3 + u_1$, $2c = u_1 + u_2$, the sides are positive numbers and the triangle inequalities are obviously satisfied.

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* Presented December 5, 1970 by D. S. MITRINOVIĆ.

[1] W. J. BLUNDON: *Inequalities associated with the triangle.* Canad. Math. Bul. **8** (1965), 615—626; see also R. W. FRUCHT: *Estudio sistematico de desigualdades de segundo grado para los radios de las circunferencias circunscrita e inscrita de un triangulo.* Scientia № **136** (1969) 114—127.

We shall express the elementary symmetric functions of u_i in terms of R, r and s. If F is the area of the triangle we have $F=rs$, $4FR=abc$, $8F^2=su_1u_2u_3$.

Therefore

$$u_1+u_2+u_3=2s, \tag{2.1}$$

$$u_1u_2u_3=8r^2s. \tag{2.2}$$

Furthermore we have

$$u_2u_3+u_3u_1+u_1u_2=\sum\{a^2-(b-c)^2\}=4\sum bc-4s^2. \tag{2.3}$$

On the other hand, from the formulas

$$r_a+r_b+r_c-r=4R,\quad rr_a=(s-b)(s-c),\ \text{etc.}, \tag{2.4}$$

where r_a, r_b, r_c are the radii of the escribed circles, we derive

$$r(4R+r)=\sum(s-b)(s-c)=\sum bc-s^2. \tag{2.5}$$

Therefore in view of 2.3 and 2.5

$$u_2u_3+u_3u_1+u_1u_2=4r(4R+r). \tag{2.6}$$

Hence the cubic equation with the roots u_i reads

$$u^3-2su^2+4r(4R+r)u-8sr^2=0, \tag{2.7}$$

(2.7) is of course well-known[2].

If R, r and s are three (positive) numbers the triangle exists if and only if (2.7) has three positive roots. But obviously, in view of the signs of its coefficients the equation has neither negative roots nor a zero root. Therefore: the necessary and sufficient condition for the existence of the triangle is: (2.7) has *real* roots.

By the substitution $u=2v+\frac{2}{3}s$ equation (2.7) is transformed into

$$v^3+pv+q=0, \tag{2.8}$$

where

$$p=-\frac{1}{3}(12Rr+3r^2-s^2),\quad q=\frac{2}{27}s(18Rr-9r^2-s^2). \tag{2.9}$$

The equation (2.8) has real roots if and only if

$$4p^3+27q^2\leqq 0. \tag{2.10}$$

Hence the condition asked for is

$$(12Rr+3r^2-s^2)^3+s^2(18Rr-9r^2-s^2)^2\leqq 0. \tag{2.11}$$

The terms s^6 and Rrs^4 vanish, the left-hand side has the factor $27r^2$ and we have: *the fundamental inequality for R, r, s is*

$$I\equiv(r^2+s^2)^2+12Rr^3-20Rrs^2+48R^2r^2-4R^2s^2+64R^3r\leqq 0. \tag{2.12}$$

[2] See f.i. A. LAISANT: *Géométrie du triangle.* Paris 1896, p. 112.

As I is a quadratic function of s^2, its zero's can be found as *irrational* functions of R and r. This gives rise to inequalities of the type $f_1(R, r) \leqq s^2 \leqq f_2(R, r)$ given by BLUNDON.

3. A geometric interpretation. In order to investigate the set of triples R, r, s satisfying (2.12) we remark that I is, of course, homogeneous polynomial and hence only the ratios of R, r and s are of interest. Therefore we introduce the variables x and y defined by

$$Rx = r, \quad Ry = s, \tag{3.1}$$

and we obtain

$$k \equiv (x^2 + y^2)^2 + 12x^3 - 20xy^2 + 48x^2 - 4y^2 + 64x \leqq 0. \tag{3.2}$$

We make use of a geometric interpretation by considering x, y as the Cartesian coordinates of a point in the Euclidean plane U. Then values x, y which satisfy (3.2) correspond to the points of a certain region G in I, lying in the first quadrant $x > 0$, $y > 0$ and bordered by the curve K with the equation $k = 0$. This equation learns us already some preliminary properties of K. *It is a quartic curve.* As K has no real intersections with the line l at infinity (in fact K is tangent to l in the isotropic points of U) it lies in a finite part of the plane. Furthermore K passes through the origin O, and, as only even powers of y appear OX is an axis of symmetry. We could obtain more information K by determining possible double-points or by making use of the fact that $k = 0$ is a quadratic equation for y^2.

We prefer however to study K by another approach. If in (2.10) equality holds equation (2.7) has two equal roots which implies that the triangle has two equal sides. Hence *the points* of K (in the first quadrant) *are the images of isosceles triangles*. If in ABC we have $AC = BC = a$ and $\angle BAC = \varphi$, then

$$AB = c = 2a\cos\varphi, \quad h_c = a\sin\varphi, \quad F = a^2\sin\varphi\cos\varphi, \quad s = a(1+\cos\varphi),$$

$$r = a\sin\varphi\cos\varphi/(1+\cos\varphi), \quad R = a/2\sin\varphi$$

and therefore

$$x = 2\cos\varphi\,(1-\cos\varphi), \quad y = 2\sin\varphi\,(1+\cos\varphi) \tag{3.3}$$

and we have arrived at a representation of K *by means of parameter* φ. The complete curve is mapped on $[0 \leqq \varphi < 2\pi]$, for the part which interests us we have $0 < \varphi < \pi/2$. By the substitution $\operatorname{tg}\frac{1}{2}\varphi = t$ we get

$$x = 4t^2(1-t^2)(1+t^2)^{-2}, \quad y = 8t(1+t^2)^{-2} \tag{3.4}$$

and the conclusion: K is a *rational* curve of the fourth order. Such a curve has always three double points. It is easy to verify that $dx/dt = dy/dt = 0$ is satisfied by $t_1 = \frac{1}{3}\sqrt{3}$, $t_2 = -\frac{1}{3}\sqrt{3}$ and $t_3 = \infty$. Hence K has three *cusps*:

$$A_1\left(\frac{1}{2}, \frac{3}{2}\sqrt{3}\right), \quad A_2\left(\frac{1}{2}, -\frac{3}{2}\sqrt{3}\right), \quad A_3(-4, 0).$$

Moreover we have $A_2A_3 = A_3A_1 = A_1A_2 = 3\sqrt{3}$ which means that *the cusps are the vertices of an equilateral triangle*. K is now recognized as a well-known curve; it is STEINER's *hypocycloid* or *deltoid*.

K is drawn in fig. 1. It passes through O and through $D(0, 2)$. Its center is $M(-1, 0)$, the three cuspidal tangents pass through it. The radii of the circumscribed and the inscribed circle are 3 and 1 respectively. As M satisfies (3.2) all points *inside* K satisfy (3.2). For our problem only points in the first quadrant are of interest. The conclusion is: *the set of points with $x>0$, $y>0$ satisfying the fundamental inequality are those of the region G, bordered by the arcs A_1O and A_1D of K and by the straight line OD.*

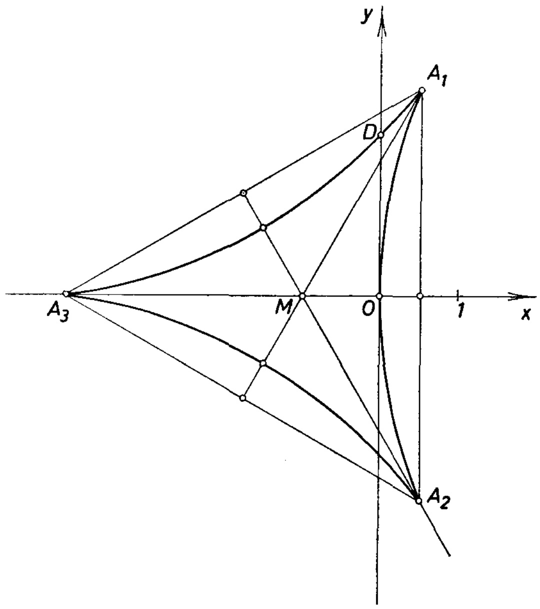

Fig. 1

The points on OD do not belong to G (they correspond to degenerated triangles) but the other points on the boundary do. A_1 is the image of the *equilateral* triangle; for points on OA_1 we have $0<\varphi\leqq\pi/3$ or $0<t\leqq\frac{1}{3}\sqrt{3}$, they correspond to isosceles triangles with a vertex angle $\gamma\geqq\pi/3$; for points on A_1D yields $\frac{\pi}{3}\leqq\varphi<\pi/2$ or $\frac{1}{3}\sqrt{3}\leqq t<1$, they are the images of isosceles triangles with $\gamma\leqq\pi/3$. It is important that K is *inside* the rectilinear triangle OA_1D and that the arcs OA_1 and A_1D *are convex*.

The region containing the points of G with the exception of A_1 will be denoted by G'.

4. Linear inequalities. The inequality (2.12) is the absolutely strongest one for R, r and s; equality holds only for isosceles triangles. All other inequalities are weaker, but they may be more elegant or more simple being of a degree less than four. Many of them are inserted in a collection recently published[3]. We remark that inequalities in which the sides a, b, c or the angles α, β, γ appear symmetrically (which holds in the majority of the cases) may be expressed as inequalities for R, r and s, in view of the fact that the elementary symmetric functions of a, b, c are given by

$$a+b+c=2s,\quad bc+ca+ab=s^2+(4R+r)\,r,\quad abc=4Rrs.$$

We deal first with some *linear* inequalities.

[3] O. BOTTEMA, R. Ž. ĐORĐEVIĆ, R. R. JANIĆ, D. S. MITRINOVIĆ, P. M. VASIĆ: *Geometric Inequalities*. Groningen 1969, further on to be quoted as GI.

The line $x=\frac{1}{2}$ passes through A_1 but all other points of G are to its left. Therefore $x\leqq\frac{1}{2}$ or

$$r\leqq\frac{1}{2}R, \tag{4.1}$$

which is EULER's inequality (GI, 5.1). Furthermore $y=\frac{3}{2}\sqrt{3}$ passes through A_1 but G_1' is below this line. Hence $y\leqq\frac{3}{2}\sqrt{3}$ or

$$s\leqq\frac{3}{2}\sqrt{3}R, \tag{4.2}$$

which is again well-known (GI, 5.3). It is clear that any line through A_1 which does not intersect the triangle OA_1D gives rise to a linear inequality. As an example we mention an inequality of JANIĆ (GI, 5.33)

$$s\sqrt{3}\leqq 5R-r, \tag{4.3}$$

which is obvious as G' is below the line $x+y\sqrt{3}-5=0$. Another example is the inequality $s\sqrt{3}\leqq(r_a+r_b+r_c)$, proved by GERRETSEN and by others (GI 5.29; 7.2), which is equivalent to

$$s\sqrt{3}\leqq 4R+r \tag{4.4}$$

(which improves (4.2) in view of (4.1)) and follows from the fact that G' is below the line $x-y\sqrt{3}+4=0$. It is easy to derive from our geometric interpretation *the strongest possible linear inequalities*: they express that G' is above the line OA_1 and below the line DA_1. They are therefore $y\geqq 3\sqrt{3}x$, or

$$s\geqq 3\sqrt{3}r, \tag{4.5}$$

which is well-known (GI, 5.11) and secondly $y\leqq(3\sqrt{3}-4)x+2$, which implies BLUNDON's inequality (GI, 5.4)

$$s\leqq 2R+(3\sqrt{3}-4)r. \tag{4.6}$$

5. Quadratic inequalities. We consider now some quadratic inequalities for R, r and s. In our geometric interpretation they mean that the region G' is inside or outside a certain *conic*. In the literature minima for s^2 in terms of R and r are well-known. They are of the general type $s^2\geqq\lambda_1 r^2+2\lambda_2 rR+\lambda_3 R^2$ or $y^2\geqq\lambda_1 x^2+2\lambda_2 x+\lambda_3$. Equality holds for a conic of which OX is an axis of symmetry. If it passes through A_1 and O we have $\lambda_3=0$ and $4\lambda_2=27-\lambda_1$.

We consider the function

$$f(x, y)=\lambda_1 x^2-y^2+\frac{1}{2}(27-\lambda_1)x. \tag{5.1}$$

If we substitute the points of K given by (3.4) and put $t^2=z$, then keeping in mind that f must have the factors z and $(3z-1)^2$, we obtain

$$f(z)=2z(1+z)^{-4}(3z-1)^2\{(\lambda_1-3)z-(\lambda_1+5)\}. \tag{5.2}$$

For the points of K in the first quadrant we have $0<z<1$. From (5.2) it follows that $f(z) \leqq 0$ if and only if $\lambda_1+5 \geqq 0$. Therefore we have proved the set of inequalities

$$s^2 \geqq \lambda_1 r^2 + \frac{1}{2}(27-\lambda_1) rR, \qquad \lambda_1 \geqq -5. \tag{5.3}$$

The *strongest inequality* of the set is that for $\lambda_1=-5$:

$$s^2 \geqq (16R-5r) r, \tag{5.4}$$

which has been derived by STEINIG (GI, 5.8; 5.17) and recognized by BLUNDON as the best one. From $R \geqq 2r$ it follows that all others are weaker. The case $\lambda_1=3$:

$$s^2 \geqq 3r(4R+r) \tag{5.5}$$

was already proved a century ago (GI, 5.5; 5.6); $\lambda_1=0$ gives the elegant inequality (GI, 5.12)

$$s^2 \geqq \frac{27}{2} Rr. \tag{5.6}$$

(5.4), (5.5) and (5.6) express that G' is respectively outside a certain ellipse, hyperbola and parabola. We shall now try to derive *maxima* for s^2, that is formulas of the type $s^2 \leqq \mu_1 r^2 + 2\mu_2 rR + \mu_3 R^2$.

If the corresponding conic $y^2=\mu_1 x^2+2\mu_2 x+\mu_3$ passes through A_1 and D we have $\mu_3=4$ and $\mu_1+4\mu_2=11$ and we meet the function

$$g(x, y) = \mu_1 x^2 - y^2 + \frac{1}{2}(11-\mu_1) x + 4. \tag{5.7}$$

If we substitute (3.4) $g(z)$ must have the factors $1-z$ and $(3z-1)^2$. We obtain

$$g(z) = 2(1+z)^{-4}(1-z)(3z-1)^2\{(1-\mu_1) z+2\} \tag{5.8}$$

and therefore $g(z) \geqq 0$ for $0<z<1$ if and only if $\mu_1 \leqq 3$. And as moreover $g(x, y) \geqq 0$ for $x=0$, $0<y<2$ we have derived the following set of inequalities

$$s^2 \leqq \mu_1 r^2 + \frac{1}{2}(11-\mu_1) rR + 4R^2, \qquad \mu_1 \leqq 3, \tag{5.9}$$

the strongest one being that with $\mu_1=3$:

$$s^2 \leqq 3r^2 + 4rR + 4R^2, \tag{5.10}$$

proved by STEINIG (GI 5.8) and *shown by* BLUNDON (GI, 5.9) *to be the best one of the type considered.*

The inequality of NAKAJIMA (GI, 7.3) reads

$$s^2 \leqq 4R^2 + \frac{11}{9}\sqrt{3} rs, \tag{5.11}$$

which means that $h(xy) = \frac{11}{9}\sqrt{3}xy - y^2 + 4 \geqq 0$ for all points of G. The curve $h(x, y)=0$ is a hyperbola H passing through A_1 and D; the inequality means that the arc of H between D and A_1 is above the arc of K between these points, (the other branch of H lies below the asymptote OX, O being the center of H). Substituting (3.4) into $h(x, y)$ we obtain

$$(5.12) \qquad h(t)=4(1+t^2)^{-4}\left\{\frac{88}{9}\sqrt{3}t^3(1-t^2)-16t^2+(1+t^2)^4\right\}.$$

As H passes through the double point $A_1\left(t=\frac{1}{3}\sqrt{3}\right)$ of K, through $D(t=1)$ and the symmetric point $D'(t=-1)$ of D with respect to O, $h(t)$ has the factors $(1-t^2)$ and $\left(t-\frac{1}{3}\sqrt{3}\right)^2$ and we have

$$(5.13) \qquad h(t)=4(1+t^2)^{-4}(1-t^2)\left(\frac{1}{3}\sqrt{3}-t\right)^2\left(3+6\sqrt{3}t-6t^2-\frac{2}{3}\sqrt{3}t^3-t^4\right),$$

which must be discussed for $\frac{1}{3}\sqrt{3}\leqq t<1$. As the last factor may be written as

$$(1-t^4)+\frac{2}{3}\sqrt{3}t(1-t^2)+6t(1-t)+\left(5\frac{1}{3}\sqrt{3}-4\right)t+2(1-t),$$

all terms of which are positive for $0<t<1$, we have indeed $h(t)\geqq 0$ and therefore (5.11) has been proved. We try to approximate the arc A_1D of K by another hyperbola J (with center O), passing trough A_1 and D and it seems attractive to choose J in such a way that it is *tangent to K at D* (which the hyperbola H is not). This tangent is $y=x+2$. The conditions give us the equation of J:

$$(5.14) \qquad j(x, y)\equiv(11-6\sqrt{3})x^2+2xy-y^2+4=0,$$

which is indeed that of a hyperbola. Substituting (3.4) into $j(x, y)$ we obtain

$$(5.15) \qquad j(t)=4(1+t^2)^{-4}\{4(11-6\sqrt{3})t^4(1-t^2)^2+16t^3(1-t^2)-16t^2+(1+t^2)^4\}.$$

As J passes through the double point $A_1\left(t=\frac{1}{3}\sqrt{3}\right)$ through $D(t=1)$ and $D'(t=-1)$, $j(t)$ must have the factors $\left(t-\frac{1}{3}\sqrt{3}\right)^2$, $(1-t)^2$ and $(1+t)$; we obtain

$$(5.16) \qquad j(t)=12(1+t^2)^{-4}(1-t)^2\left(t-\frac{1}{3}\sqrt{3}\right)^2$$
$$\times\{1+(1+2\sqrt{3})t+(-1+2\sqrt{3})t^2+(15-8\sqrt{3})t^3\}$$

and therefore

$$j(t)\geqq 0, \qquad 0<t<1,$$

with equality only for $t=\frac{1}{3}\sqrt{3}$. *Hence we have proved the inequality*

$$(5.17) \qquad s^2\leqq 4R^2+2rs+(11-6\sqrt{3})r^2,$$

which seems to be new. It is stronger than NAKAJIMA's: the difference of the right hand side of (5.11) and that

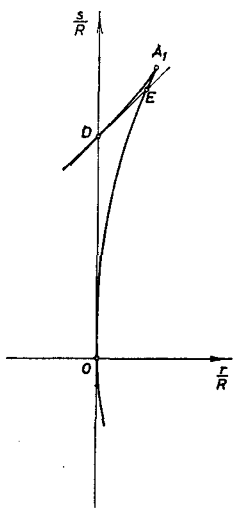

Fig. 2

of (5.17) is $\frac{1}{9}(11\sqrt{3}-18)\,r\,(s-3\sqrt{3}r)$, which is $\geqq 0$ in view of (4.5). The arc A_1D of J is between that of K and H.

6. Higher order inequalities. These may be treated in the same way; the method comes to this that we have to determine the position of region G with respect to a certain curve of higher order in the image plane. We give two examples.

In his paper on trigonometric inequalities BAGER[4] has stated the inequality

(6.1) $$\sum \cos\beta\cos\gamma \geqq \frac{9}{4}\sqrt{3}\prod \cot\alpha$$

as a conjecture. If we translate it by means of

(6.2) $$\sum \cos\beta\cos\gamma = (r^2+s^2-4R^2)/4R^2, \quad \prod \cot\alpha = \{s^2-(2R+r)^2\}/2rs$$

it reads

(6.3) $$rs\,(r^2+s^2-4R^2)-\frac{9}{2}\sqrt{3}R^2\{s^2-(2R+r)^2\} \geqq 0,$$

or

(6.4) $$l(x, y) \equiv xy\,(x^2+y^2-4)+\frac{9}{2}\sqrt{3}\,(x^2-y^2+4x+4) \geqq 0,$$

which is an inequality of the fourth order. The quartic curve L, with equation $l=0$, passes through D and A_1. A further discussion shows that the points of L which lie in the strip $0 \leqq x \leqq \frac{1}{2}$ are those of three arcs, the arc b from D to A_1, a second one above b (with OY as an asymptote) and a third below OX. Hence we must show that the arc D_1A of K is below b. Substituting (3.4) into $l(x, y)$ we obtain

(6.5) $$l_1 = xy\,(x^2+y^2-4) = 2^7(1+t^2)^{-7}\,t^3(1-t^2)^2(-3t^4+12t^2-1)$$

and

(6.6) $$l_2 = x^2-y^2+4x+4 = 4\,(1+t^2)^{-4}(1-t^2)^3(t^4-6t^2+1).$$

Keeping in mind that L passes through the double point $A_1\left(t=\frac{1}{3}\sqrt{3}\right)$ of K we get after some algebra

(6.7) $$l(t) = 6\,(1+t^2)^{-7}(1-t^2)^2\left(t-\frac{1}{3}\sqrt{3}\right)^2$$
$$\times(3\sqrt{3}t^8+6t^7-6\sqrt{3}t^6-78t^5-92\sqrt{3}t^4+98t^3+54\sqrt{3}t^2+54t+9\sqrt{3}).$$

The last factor may be written as

(6.8) $$3\sqrt{3}t^8+6t^7+6\sqrt{3}(1-t^6)+78t^3(1-t^2)+54\sqrt{3}t^2(1-t^2)$$
$$+54t\,(1-t^3)+20t^3(1-t)+2\,(37-19\sqrt{3})\,t^4+3\sqrt{3},$$

from which it follows that it is positive for $0<t<1$. Hence $l(t) \geqq 0$ for $0 \leqq t \leqq 1$ and the conjecture (6.1) has been proved. It is a strong inequality

[4] A. BAGER: *A family of goniometric inequalities.* Same Publications, pp 5—25.

for it can be shown that the arc b is below the straight line DA_1. Therefore although L remains outside G' it penetrates the rectilinear triangle ODA_1.

We consider another inequality, which is a counterpart of (6.1). Similar to (6.2) we have

(6.9) $$\prod \operatorname{tg}\frac{1}{2}\alpha = \frac{r}{s},$$

and, as a consequence of (5.10), the known inequality

(6.10) $$\prod \operatorname{tg}\frac{1}{2}\alpha \geqq \prod \cot \alpha,$$

which is a reason to compare $\sum \cos\beta\cos\gamma$ with $\frac{9}{4}\sqrt{3}\prod \operatorname{tg}\frac{1}{2}\alpha$.

We shall prove BAGER's inequality[5]

(6.11) $$\frac{9}{4}\sqrt{3}\prod \operatorname{tg}\frac{1}{2}\alpha - \sum \cos\beta\cos\gamma \geqq 0,$$

that is the cubic inequality

(6.12) $$9\sqrt{3}R^2r - s(r^2+s^2-4R^2) \geqq 0$$

or

(6.13) $$m(xy) \equiv 9\sqrt{3}x - (x^2+y^2-4)y \geqq 0.$$

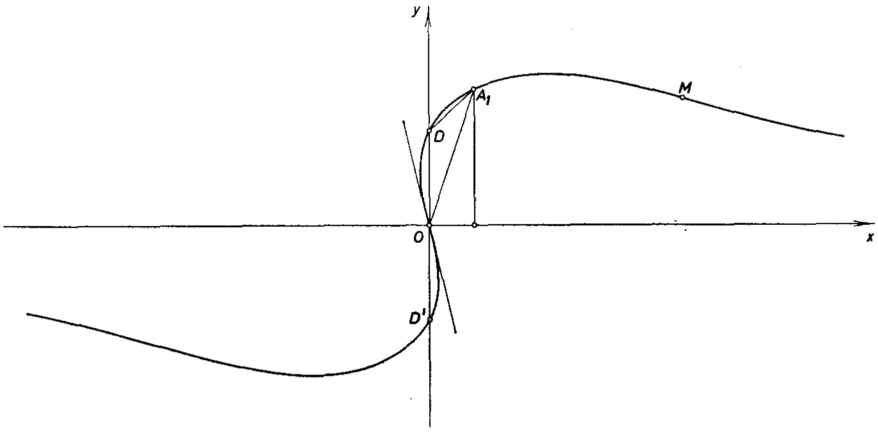

Fig. 3

Just for a change we do it graphically. The curve M given by $m(xy)=0$ is symmetric with respect to O, it is a circular curve, its only real asymptote is OX, y is positive for large values of x, M passes through A_1 (the tangent at A_1 has the positive slope $\frac{5}{11}\sqrt{3}$, which is less than of DA_1), through D (the tangent here has the slope $\frac{9}{8}\sqrt{3}$ which is greater than that of DA_1),

[5] A. BAGER: *A family of goniometric inequalities*. Same Publications, pp. 5—25.

through O $\left(\text{where it has an inflexion point, the tangent being } y=-\frac{9}{4}\sqrt{3}x\right)$ and through D'. The third intersection of M and DA_1 is outside DA_1. From all this we can easily make a scetch of M (fig. 3). The region G' is below M and (6.13) follows. It is remarkable in so far that M passes through the *three* vertices of G; equality holds for the equilateral and for both isosceles degenerated triangles. On the other hand it is not a very strong inequality, for M does not even enter the rectilinear triangle ODA_1.

7. A final remark. Some known inequalities hold for acute (or for obtuse) triangles only. It is well-known that a triangle is acute, right-angled or obtuse if $s-r-2R$ is negative, zero or positive respectively. In the image plane U the equation $y-x-2=0$ represents the straight line n through D and the point $E(\sqrt{2}-1, \sqrt{2}+1)$ on the arc OA_1 of K, corresponding to

$$t=\sqrt{2}-1=\operatorname{tg}\frac{\pi}{8}$$

and being the image of the rectangular isosceles triangle. The line n divides the region G in the subregion G_1, bordered by OD, n and the arc OE, and the subregion G_2, bordered by n and the arcs DA_1 and EA_1. Points of G_1 (G_2) correspond to obtuse (acute) triangles, as BLUNDON remarked already. The mapping might be used to prove inequalities which hold for acute (or obtuse) triangles (fig. 2).

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