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339. A FAMILY OF GONIOMETRIC INEQUALITIES*

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Introduction

1. The present paper is inspired by my study of the first eight chapters, in particular chapter two, of the book [GI]. All decimal references will be to this book, in which a sufficient number of refeiences to more original literature can be found.

The main part of the paper provides a graphical representation of about 300 goniometric inequalities of a certain form (see 5). Of those inequalities we need only prove 38 as the remaining will follow by transitivity. Unfortunately, I have only been able to prove 35 of the fundamental inequalities; three have been stated as conjectures.

For 32 of the fundamental inequalities I have found quite elementary proofs. To make this part of the paper accessible to a wide circle of readers I have provided this rather long introduction, which contains nothing new.

The final part of the paper consists of various applications of the inequalities in the (proved part of the) graph. Of the three more „advanced" fundamental inequalities I have made only a single application.

2. We shall only derive results valid for an arbitrary triangle ABC (it may be a very good exercise for the reader to develop inequalities valid for acute triangles only!). In accordance with [GI] the sides will be denoted a, b, c, the angles α, β, γ, the area F, the semiperimeter s, the circum-radius R, the inradius r, the exradii r_a, r_b, r_c, the circum-center O and the incenter I.

We deviate a bit from [GI] in using three-letter denotations for all six goniometric functions considered: sin, cos, tan, cot, csc, sec. The reader may translate from [GI] to the present paper by means of tg = tan, cotg = cot and cosec = csc.

If either $\alpha \leqq \beta \leqq \gamma$ or $\alpha \geqq \beta \geqq \gamma$ we shall call β the middle angle.

3. For any given triangle there will exist a triangle with angles $\frac{\pi}{2}-\frac{\alpha}{2}$, $\frac{\pi}{2}-\frac{\beta}{2}$, $\frac{\pi}{2}-\frac{\gamma}{2}$. This transformation of angles will be denoted σ. If the given triangle

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is acute there will exist a triangle with angles $\pi-2\alpha$, $\pi-2\beta$, $\pi-2\gamma$. This transformation of angles will be denoted τ. The reader may observe that σ and τ, in a certain sense, are inverse to each other. From every generally valid equation or inequality of angles we may immediately form another by using σ. When we restrict our attention to acute triangles (which we will be forced to in some proofs) τ can be used in the same manner.

4. Let f be a function defined on the open interval $(0, \pi)$. We shall use the abbreviations

$$\sum f(\alpha)=f(\alpha)+f(\beta)+f(\gamma),$$
$$\sum f(\beta)f(\gamma)=f(\beta)f(\gamma)+f(\gamma)f(\alpha)+f(\alpha)f(\beta),$$
$$\prod f(\alpha)=f(\alpha)f(\beta)f(\gamma).$$

In a few cases we shall use the similar abbreviations $\sum f(a)$, $\sum f(b)f(c)$ and $\prod f(a)$, where now f is supposed to be defined on the positive real axis.

5. The functions $\sin x$, $\sin\frac{x}{2}$, $\cos x$, $\cos\frac{x}{2}$, $\tan\frac{x}{2}$, $\cot x$, $\cot\frac{x}{2}$, $\csc x$, $\csc\frac{x}{2}$, $\sec\frac{x}{2}$ are all defined on $(0, \pi)$. Hence they give rise to 30 symmetric functions of the three vaiiables α, β, γ (or of the triangle ABC) as described in 4. Properly speaking, though, there are only 26 distinct such functions, among those the constant 1 (see 8). Each of these 26 functions we shall normalize by multiplying it with a suitable positive constant in order to make the function assume the value 1 at $\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right)$, i.e. for equilateral triangles. Examples of such normalized functions are: $\frac{\sqrt{3}}{3}\sum\cot\alpha$, $\frac{4}{9}\sum\sin\beta\sin\gamma$ and $8\prod\sin\frac{\alpha}{2}$.

Our goal shall be to find all inequalities

$$\Phi(\alpha, \beta, \gamma)\leqq\Psi(\alpha, \beta, \gamma)$$

between normalized functions Φ and Ψ, valid for all triangles and being equations if and only if the triangle is equilateral.

Of course the inequalities will very rarely appear in their normalized form; two inequalities, one of which can be derived from the other by multiplication by a positive number, will be identified.

6. Whenever possible we shall adhere to the following convention. When a trigonometric inequality using the sign „$\leqq$" or the sign „$\geqq$" is stated, equality will obtain in the equilateral case and in this only. The few exceptions will be clearly indicated. In all proofs the discussion of the condition for equality is easy and will be left to the reader.

7. We shall use a few algebraic inequalities concerning a triple (x, y, z) of arbitrary non-negative numbers. We shall state the inequalities without proofs (which will easily be found elsewhere): The arithmetic-geometric mean inequality

$$\frac{x+y+z}{3}\geqq\sqrt[3]{xyz},$$

and the arithmetic-square mean inequality

$$x+y+z \leqq \sqrt{3(x^2+y^2+z^2)},$$

which we shall also use in one of the alternative forms

$$yz+zx+xy \leqq x^2+y^2+z^2,$$

$$3(yz+zx+xy) \leqq (x+y+z)^2.$$

The last inequality, in which x, y, z are restricted to positive values, is the arithmetic-harmonic mean inequality

$$(x+y+z)(x^{-1}+y^{-1}+z^{-1}) \geqq 9.$$

In each of the inequalities stated above the condition of equality is $x=y=z$.

8. We shall use some goniometric identities, which we shall now state and prove presupposing only very elementary trigonometry (among this the most elementary geometry of the incircle and the excircles). For later reference the formulas will be indicated by capital letters (numbers being reserved for a certain purpose, see 11).

If no angle is $\frac{\pi}{2}$ we have

$$-\tan\gamma = \tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta},$$

which can be reduced to

$$\sum \tan\alpha = \prod \tan\alpha.$$

Using σ (see 3) on this we get our first formula:

(A) $$\sum \cot\frac{\alpha}{2} = \prod \cot\frac{\alpha}{2}.$$

Multiplying (A) and the formula preceding it by $\prod \tan\frac{\alpha}{2}$ and $\prod \cot\alpha$, respectively, we get the formulas

(B) $$\sum \tan\frac{\beta}{2}\tan\frac{\gamma}{2} = 1,$$

(C) $$\sum \cot\beta\cot\gamma = 1$$

(it is easily verified that (C) remains valid in right triangles, excluded in the proof given). Next we shall prove:

(D) $$\sum \sin\alpha = 4\prod \cos\frac{\alpha}{2}.$$

Proof. $$(\sin\alpha+\sin\beta)+\sin\gamma = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}$$

$$= 2\cos\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2}+2\cos\frac{\alpha+\beta}{2}\cos\frac{\gamma}{2}$$

$$= 2\cos\frac{\gamma}{2}\left(\cos\frac{\alpha-\beta}{2}+\cos\frac{\alpha+\beta}{2}\right) = 4\prod \cos\frac{\alpha}{2}.$$

(E) $$\sum \cos \alpha = 1 + 4 \prod \sin \frac{\alpha}{2}.$$

Proof. $(\cos \alpha + \cos \beta) - (1 - \cos \gamma) = 2 \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} - 2 \sin^2 \frac{\gamma}{2}$

$$= 2 \sin \frac{\gamma}{2} \left(\cos \frac{\alpha-\beta}{2} - \cos \frac{\alpha+\beta}{2} \right) = 4 \prod \sin \frac{\alpha}{2}.$$

(F) $$\sum \cos 2\alpha = -1 - 4 \prod \cos \alpha.$$

Proof. $(1 + \cos 2\alpha) + (\cos 2\beta + \cos 2\gamma) = 2 \cos^2 \alpha + 2 \cos (\beta + \gamma) \cos (\beta - \gamma)$

$$= -2 \cos \alpha (\cos (\beta + \gamma) + \cos (\beta - \gamma)) = -4 \prod \cos \alpha.$$

As $\cos^2 x = \frac{1 + \cos 2x}{2}$ and $\sin^2 x = \frac{1 - \cos 2x}{2}$,

(F) yields the next two formulas:

(G) $$\sum \cos^2 \alpha = 1 - 2 \prod \cos \alpha,$$

(H) $$\sum \sin^2 \alpha = 2 (1 + \prod \cos \alpha).$$

Using σ on (G) we get:

(I) $$\sum \sin^2 \frac{\alpha}{2} = 1 - 2 \prod \sin \frac{\alpha}{2}.$$

The next formula we shall prove is:

(J) $$2 \sum \cot \alpha = \frac{\sum \sin^2 \alpha}{\prod \sin \alpha}.$$

Proof. $2 \sum \cot \alpha = \sum (\cot \beta + \cot \gamma) = \sum \frac{\sin (\beta + \gamma)}{\sin \beta \sin \gamma} = \sum \frac{\sin \alpha}{\sin \beta \sin \gamma} = \frac{\sum \sin^2 \alpha}{\prod \sin \alpha}.$

Combining (H) and (J) we get

(K) $$\sum \cot \alpha = \frac{1 + \prod \cos \alpha}{\prod \sin \alpha},$$

and from this by σ:

(L) $$\sum \tan \frac{\alpha}{2} = \frac{1 + \prod \sin \frac{\alpha}{2}}{\prod \cos \frac{\alpha}{2}}.$$

Dividing (D) by $\frac{1}{2} \prod \sin \alpha = 4 \prod \left(\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \right)$ we get:

(M) $$2 \sum \csc \beta \csc \gamma = \prod \csc \frac{\alpha}{2}.$$

Multiplying (L) by $\prod \cot \frac{\alpha}{2}$ we find:

(N) $$\sum \cot \frac{\beta}{2} \cot \frac{\gamma}{2} = 1 + \prod \csc \frac{\alpha}{2}.$$

In one case we shall need the less well-known formula

(O) $$\sum \cot^2 \frac{\alpha}{2} = \left(\sum \cot \frac{\alpha}{2}\right)\left(4\sum \cot \alpha - \sum \cot \frac{\alpha}{2}\right).$$

Proof. As $a = 2R\sin\alpha$ and $F = 2R^2 \prod \sin\alpha$, formula (J) can restated as $2\sum \cot\alpha = (\sum a^2)/(2F)$. Hence

$$\sum \cot^2 \frac{\alpha}{2} + \left(\sum \cot \frac{\alpha}{2}\right)^2 = \frac{(s-a)^2+(s-b)^2+(s-c)^2+s^2}{r^2} = \frac{a^2+b^2+c^2}{r^2}$$

$$= \frac{4F\sum \cot\alpha}{r^2} = \frac{4s}{r}\sum \cot\alpha = 4\left(\sum \cot \frac{\alpha}{2}\right)\left(\sum \cot\alpha\right).$$

That there are only 26 normalized functions (see 5) follows from formulas (A), (B), (C), (D) and (M). At the same time we see that the constant function 1 is one of the functions.

9. We shall now derive a few formulas of another kind. Dividing

$$(s-a)+(s-b)+(s-c) = s \quad \text{by} \quad F = rs = r_a(s-a) = r_b(s-b) = r_c(s-c)$$

we get the formula

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c} = \frac{1}{r}.$$

The next formula is

$$r_a + r_b + r_c - r = 4R.$$

Proof. $$r_a + r_b + r_c - r = F\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}\right)$$

$$= F\left(\frac{c}{(s-a)(s-b)}+\frac{c}{s(s-c)}\right) = \frac{Fabc}{s(s-a)(s-b)(s-c)} = \frac{abc}{F} = 4R.$$

By means of this formula we get

$$r(4R+r) = \sum rr_a = \sum (s-b)\tan\frac{\beta}{2}(s-c)\cot\frac{\beta}{2} = \sum (s-b)(s-c)$$

$$= \sum s^2 - s\sum (b+c) + \sum bc = 3s^2 - 4s^2 + \sum bc.$$

Hence:

$$bc + ca + ab = s^2 + r(4R+r).$$

As $\sum a^2 = (\sum a)^2 - 2\sum bc$ this formula yields the following:

$$a^2+b^2+c^2 = 2[s^2 - r(4R+r)].$$

This is all we shall need but we shall give the following remark: From the last formula but one combined with $a+b+c=2s$ and $abc = 4RF = 4Rrs$ we conclude that, in fact, every symmetric polynomial in a, b, c can be expressed as a polynomial in s, R, r.

10. As we shall see now even some symmetric goniometric functions of α, β, γ can be expressed as (rational) functions of s, R, r. By means of the formulas $\sum a = 2R\sum \sin\alpha$ and (D) we get the three formulas:

$$\sum \sin\alpha = \frac{s}{R}, \quad \prod \cos\frac{\alpha}{2} = \frac{s}{4R}, \quad \prod \sec\frac{\alpha}{2} = \frac{4R}{s}.$$

As $F=2R^2\prod \sin\alpha$ and $\prod \sin\frac{\alpha}{2}=\frac{1}{8}(\prod \sin\alpha)\left(\prod \sec\frac{\alpha}{2}\right)$ we get:

$$\prod \sin\alpha=\frac{rs}{2R^2},\quad \prod \csc\alpha=\frac{2R^2}{rs},\quad \prod \sin\frac{\alpha}{2}=\frac{r}{4R},\quad \prod \csc\frac{\alpha}{2}=\frac{4R}{r}.$$

The formulas (M) and (E) yield:

$$\sum \csc\beta\csc\gamma=\frac{2R}{r},\quad \sum \cos\alpha=\frac{R+r}{R}.$$

By the formulas $\sum \tan\frac{\alpha}{2}=\left(\prod \sin\frac{\alpha}{2}\right)\left(\prod \sec\frac{\alpha}{2}\right)$ and (A) we get:

$$\prod \tan\frac{\alpha}{2}=\frac{r}{s},\quad \sum \cot\frac{\alpha}{2}=\prod \cot\frac{\alpha}{2}=\frac{s}{r}.$$

The formulas (L) and (N) yield:

$$\sum \tan\frac{\alpha}{2}=\frac{4R+r}{s},\quad \sum \cot\frac{\beta}{2}\cot\frac{\gamma}{2}=\frac{4R+r}{r}.$$

The formula $2\sum \cot\alpha=(\sum a^2)/(2F)$ (see the proof of (O)) together with the last formula in 9 yield the formula

$$\sum \cot\alpha=\frac{s^2-r(4R+r)}{2rs}.$$

As

$$\sum \sin\beta\sin\gamma=(\sum bc)/(4R^2),\quad \sum \csc\alpha=(\sum \sin\beta\sin\gamma)(\prod \csc\alpha)$$

and

$$\sum \cos\beta\cos\gamma=\sum \sin\beta\sin\gamma-\sum \cos\alpha$$

we get, by means of a formula from 9, the following three formulas:

$$\sum \sin\beta\sin\gamma=\frac{s^2+r(4R+r)}{4R^2},$$

$$\sum \csc\alpha=\frac{s^2+r(4R+r)}{2rs},$$

$$\sum \cos\beta\cos\gamma=\frac{r^2+s^2-4R^2}{4R^2}.$$

By formula (F) we have

$$(1+\sum \cos\alpha)^2-(\sum \sin\alpha)^2$$
$$=1+\sum \cos^2\alpha+2\sum \cos\alpha+2\sum \cos\beta\cos\gamma-\sum \sin^2\alpha-2\sum \sin\beta\sin\gamma$$
$$=1+\sum \cos^2\alpha-\sum \sin^2\alpha=1+\sum \cos 2\alpha=-4\prod \cos\alpha.$$

Inserting here the expressions for $\sum \cos\alpha$ and $\sum \sin\alpha$ found above we get:

$$\prod \cos\alpha=\frac{s^2-(2R+r)^2}{4R^2}.$$

As $\prod \cot\alpha=(\prod \cos\alpha)(\prod \csc\alpha)$ we finally get:

$$\prod \cot\alpha=\frac{s^2-(2R+r)^2}{2rs}.$$

The graph of inequalities

11. We shall now prove the 35 inequalities and state the three conjectures (see 1). The 35 fundamental (proved) inequalities will be indicated by numbers, by means of which the reader may locate them in the graph. As lemmas for our proofs, we need some further inequalities and two theorems; these lemmas will be indicated by small letters. The first and by far the most fundamental is EULER's inequality

(a) $$R \geqq 2r.$$

It is a corollary of the formula

$$OI^2 = R^2 - 2Rr,$$

but we shall indicate the following more elementary proof: By the arithmetic--harmonic mean inequality (see 7) we have

$$(r_a + r_b + r_c)(r_a^{-1} + r_b^{-1} + r_c^{-1}) \geqq 9.$$

By means of two formulas from 9 this reduces to

$$(4R + r)r^{-1} \geqq 9,$$

which immediately yields (a).

12. By (a) and $\prod \sin \frac{\alpha}{2} = \frac{r}{4R}$ (see 10) we get

(b) $$\prod \sin \frac{\alpha}{2} \leqq \frac{1}{8}.$$

By (b) and formula (N) we get our first fundamental inequality:

(1) $$\sum \cot \frac{\beta}{2} \cot \frac{\gamma}{2} \leqq \frac{9}{8} \prod \csc \frac{\alpha}{2}.$$

Considering first the acute case we get, using τ on (b):

(c) $$\prod \cos \alpha \leqq \frac{1}{8},$$

which is trivially seen to be true in the other cases. By (c) we get:

$$1 + \prod \cos \alpha \leqq \frac{9}{8}.$$

Dividing this by $\prod \sin \alpha$ and using formula (K) we get:

(2) $$\sum \cot \alpha \leqq \frac{9}{8} \prod \csc \alpha.$$

Using σ on (2) we get:

(3) $$\sum \tan \frac{\alpha}{2} \leqq \frac{9}{8} \prod \sec \frac{\alpha}{2}.$$

13. From (b) and formula (E) we get:

(d) $$\sum \cos \alpha \leqq \frac{3}{2}.$$

Using σ on (d) we get:

(4) $$\sum \sin \frac{\alpha}{2} \leqq \frac{3}{2}.$$

By the arithmetic-harmonic mean inequality we get:

$$\left(\sum \sin \frac{\alpha}{2}\right)\left(\sum \csc \frac{\alpha}{2}\right) \geqq 9.$$

Hence by (4):

(5) $$\sum \csc \frac{\alpha}{2} \geqq 6.$$

Dividing (4) by $\prod \sin \frac{\alpha}{2}$ we get:

(6) $$\sum \csc \frac{\beta}{2} \csc \frac{\gamma}{2} \leqq \frac{3}{2} \prod \csc \frac{\alpha}{2}.$$

By (6) and the arithmetic-harmonic mean inequality we get:

(7) $$\sum \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \geqq 6 \prod \sin \frac{\alpha}{2}.$$

In the acute case we get using τ on (7):

(8) $$\sum \cos \beta \cos \gamma \geqq 6 \prod \cos \alpha,$$

which is trivially seen to be true in the right case. Considering, finally, the obtuse case, we shall suppose that $\gamma > \frac{\pi}{2}$. By division by the negative number $\prod \cos \alpha$ we rewrite (8) as

$$\sec \alpha + \sec \beta < 6 + \sec (\alpha + \beta).$$

As $\alpha + \beta < \frac{\pi}{2}$ we cannot have both $\cos \alpha \leqq \frac{1}{6}$ and $\cos \beta \leqq \frac{1}{6}$. Hence we suppose that $\cos \alpha > \frac{1}{6}$. Then $\sec \alpha < 6$ and $\sec \beta < \sec (\alpha + \beta)$ (as sec is increasing), from which the rewritten inequality follows by addition.

14. Using (4), (b) and formula (I) we get:

$$2 \sum \sin \frac{\beta}{2} \sin \frac{\gamma}{2} = \left(\sum \sin \frac{\alpha}{2}\right)^2 + 2 \prod \sin \frac{\alpha}{2} - 1 \leqq \frac{3}{2},$$

(e) $$\sum \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \leqq \frac{3}{4}.$$

By means of (e) we shall prove the inequality

(f) $$2 \sum \cos \alpha \geqq \sum \cos (\beta - \gamma).$$

Proof. $2\sum\cos\alpha-\sum\cos(\beta-\gamma)=\sum[\cos\beta+\cos\gamma-\cos(\beta-\gamma)]$

$$=\sum\left[2\cos\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2}-2\cos^2\frac{\beta-\gamma}{2}+1\right]$$

$$=\sum\left[1+2\cos\frac{\beta-\gamma}{2}\left(\cos\frac{\beta+\gamma}{2}-\cos\frac{\beta-\gamma}{2}\right)\right]$$

$$=3-4\sum\cos\frac{\beta-\gamma}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\geqq 3-4\sum\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\geqq 0.$$

By (f) we get:

$$2\sum\sin\beta\sin\gamma=\sum\cos(\beta-\gamma)+\sum\cos\alpha\leqq 3\sum\cos\alpha,$$

(9) $$\sum\sin\beta\sin\gamma\leqq\frac{3}{2}\sum\cos\alpha.$$

Using σ on (9) we get:

(10) $$\sum\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\leqq\frac{3}{2}\sum\sin\frac{\alpha}{2}.$$

15. By the arithmetic-square mean inequality, formula (H), and (c) we get:

$$\sum\sin\alpha\leqq\sqrt{3\sum\sin^2\alpha}=\sqrt{6\left(1+\prod\cos\alpha\right)}\leqq\frac{3\sqrt{3}}{2}.$$

Thus we have proved the inequality

(g) $$\sum\sin\alpha\leqq\frac{3\sqrt{3}}{2},$$

on which we use σ and find:

(11) $$\sum\cos\frac{\alpha}{2}\leqq\frac{3\sqrt{3}}{2}.$$

Dividing (11) by $\prod\cos\frac{\alpha}{2}$ we get:

(12) $$\sum\sec\frac{\beta}{2}\sec\frac{\gamma}{2}\leqq\frac{3\sqrt{3}}{2}\prod\sec\frac{\alpha}{2}.$$

By (11) and the arithmetic-harmonic mean inequality we get:

(13) $$\sum\sec\frac{\alpha}{2}\geqq 2\sqrt{3}.$$

We multiply (13) by $\prod\cos\frac{\alpha}{2}$ and get:

(14) $$\sum\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\geqq 2\sqrt{3}\prod\cos\frac{\alpha}{2}.$$

16. Combining (10) and (14) we get:

$$\sum\sin\frac{\alpha}{2}\geqq\frac{4}{\sqrt{3}}\prod\cos\frac{\alpha}{2}.$$

Dividing this by $\prod\sin\frac{\alpha}{2}$ we find:

(15) $$\sum\csc\frac{\beta}{2}\csc\frac{\gamma}{2}\geqq\frac{4}{\sqrt{3}}\prod\cot\frac{\alpha}{2}.$$

17. By means of (g) and formula (D) we get

(h) $$\prod \cos\frac{\alpha}{2} \leqq \frac{3\sqrt{3}}{8}.$$

We multiply (h) by $8\prod \csc\alpha$ and get:

(16) $$\prod \csc\frac{\alpha}{2} \leqq 3\sqrt{3}\prod \csc\alpha.$$

By (g) and the arithmetic-square mean inequality (see 7) we get:

$$3\sum \sin\beta\sin\gamma \leqq \left(\sum \sin\alpha\right)^2 \leqq \frac{3\sqrt{3}}{2}\sum \sin\alpha,$$

(17) $$\sum \sin\beta\sin\gamma \leqq \frac{\sqrt{3}}{2}\sum \sin\alpha.$$

Using σ on (17) we get:

(18) $$\sum \cos\frac{\beta}{2}\cos\frac{\gamma}{2} \leqq \frac{\sqrt{3}}{2}\sum \cos\frac{\alpha}{2}.$$

Dividing this by $\prod \cos\frac{\alpha}{2}$ we find:

(19) $$\sum \sec\frac{\alpha}{2} \leqq \frac{\sqrt{3}}{2}\sum \sec\frac{\beta}{2}\sec\frac{\gamma}{2}.$$

We multiply (h) by $8\prod \sin\frac{\alpha}{2}$ and $\prod \tan\frac{\alpha}{2}$, respectively, and get the following two inequalities:

(20) $$\prod \sin\alpha \leqq 3\sqrt{3}\prod \sin\frac{\alpha}{2},$$

(21) $$\prod \sin\frac{\alpha}{2} \leqq \frac{3\sqrt{3}}{8}\prod \tan\frac{\alpha}{2}.$$

18. By formula (D) the first inequality in the proof of (17) can be rewritten as

$$3\sum \sin\beta\sin\gamma \leqq 16\prod \cos^2\frac{\alpha}{2}.$$

Dividing this by $3\prod \sin\alpha$ we get:

(22) $$\sum \csc\alpha \leqq \frac{2}{3}\prod \cot\frac{\alpha}{2}.$$

19. By the arithmetic-harmonic mean inequality we have

$$\left(\sum \sin\alpha\right)\left(\sum \csc\alpha\right) \geqq 9.$$

Dividing by $\sum \sin\alpha$ and using formula (D) we get:

(23) $$\sum \csc\alpha \geqq \frac{9}{4}\prod \sec\frac{\alpha}{2}.$$

We multiply (23) by $\prod \sin\alpha$ and get:

(24) $$\sum \sin\beta\sin\gamma \geqq 18\prod \sin\frac{\alpha}{2}.$$

20. Combining (22) and (23) we get:

$$\prod \cot \frac{\alpha}{2} \geqq \frac{27}{8} \prod \sec \frac{\alpha}{2},$$

which we restate as

(25) $$\prod \tan \frac{\alpha}{2} \leqq \frac{8}{27} \prod \cos \frac{\alpha}{2}.$$

In the acute case we get by τ:

(26) $$\prod \cot \alpha \leqq \frac{8}{27} \prod \sin \alpha,$$

and this is trivial in the other cases.

21. As $2 \sum \cos \alpha = \sum (\cos \beta + \cos \gamma) = 2 \sum \sin \frac{\alpha}{2} \cos \frac{\beta-\gamma}{2} \leqq 2 \sum \sin \frac{\alpha}{2}$ we have:

(27) $$\sum \cos \alpha \leqq \sum \sin \frac{\alpha}{2}.$$

The reader may note that the same idea can be used to give a much simpler proof of 2.4 than the one offered in [GI].

22. Let x, y, z be real numbers satisfying $x+y+z=0$. Then we have

$$\sin x + \sin y + \sin z = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2} + 2 \sin \frac{z}{2} \cos \frac{z}{2}$$

$$= -2 \sin \frac{z}{2} \cos \frac{x-y}{2} + 2 \sin \frac{z}{2} \cos \frac{x+y}{2}$$

$$= 2 \sin \frac{z}{2} \left(\cos \frac{x+y}{2} - \cos \frac{x-y}{2} \right) = -4 \sin \frac{x}{2} \sin \frac{y}{2} \sin \frac{z}{2}.$$

As $\sum \left(\frac{\pi}{3} - \alpha \right) = 0$ the formula just proved yields:

$$\frac{\sqrt{3}}{2} \sum \cos \alpha - \frac{1}{2} \sum \sin \alpha = \sum \sin \left(\frac{\pi}{3} - \alpha \right) = -4 \prod \sin \left(\frac{\pi}{6} - \frac{\alpha}{2} \right).$$

From this formula we easily derive the following theorem:

(i) Let β denote the middle angle (see 2). Then:

$$\beta > \frac{\pi}{3} \Rightarrow \sum \sin \alpha < \sqrt{3} \sum \cos \alpha,$$

$$\beta = \frac{\pi}{3} \Rightarrow \sum \sin \alpha = \sqrt{3} \sum \cos \alpha,$$

$$\beta < \frac{\pi}{3} \Rightarrow \sum \sin \alpha > \sqrt{3} \sum \cos \alpha.$$

Using σ on (i) we get the parallel theorem:

(j) Let β denote the middle angle. Then:

$$\beta < \frac{\pi}{3} \Rightarrow \sum \cos\frac{\alpha}{2} > \sqrt{3}\sum \sin\frac{\alpha}{2},$$
$$\beta = \frac{\pi}{3} \Rightarrow \sum \cos\frac{\alpha}{2} = \sqrt{3}\sum \sin\frac{\alpha}{2},$$
$$\beta > \frac{\pi}{3} \Rightarrow \sum \cos\frac{\alpha}{2} < \sqrt{3}\sum \sin\frac{\alpha}{2}.$$

23. The two theorems from 22 make it possible for us to prove the inequality

(28) $$\sum \cos\frac{\alpha}{2} \geqq \sqrt{3}\sum \cos\alpha.$$

Proof. We suppose that $\alpha \leqq \beta \leqq \gamma$ and consider three cases separately

1) $\alpha = \beta = \gamma = \frac{\pi}{3}$. Clearly equality holds.

2) $\alpha < \beta \leqq \frac{\pi}{3} < \gamma$. By (j) and (27) we get

$$\sum \cos\frac{\alpha}{2} \geqq \sqrt{3}\sum \sin\frac{\alpha}{2} > \sqrt{3}\sum \cos\alpha.$$

3) $\alpha < \frac{\pi}{3} < \beta < \gamma$. By (14), (18), and (i) we have

$$\sum \cos\frac{\alpha}{2} > \sum \sin\alpha > \sqrt{3}\sum \cos\alpha.$$

Thus the proof is finished.

24. The next inequality

(29) $$2\sum \sin\frac{\beta}{2}\sin\frac{\gamma}{2} \leqq \sum \cos\alpha$$

is proved in 2.56 and I know no other proof. I shall reproduce the proof. In this we disregard the convention from 6, but the reader will observe that we stick to the convention in (29). We let β be the middle angle and choose the notations α and γ such that either $\alpha \leqq \frac{\pi}{3} \leqq \beta \leqq \gamma$ or $\alpha \geqq \frac{\pi}{3} \geqq \beta \geqq \gamma$. Then, clearly:

$$\left(2\sin\frac{\beta}{2} - 1\right)\left(2\sin\frac{\gamma}{2} - 1\right) \geqq 0$$

$$2\sin\frac{\beta}{2} + 2\sin\frac{\gamma}{2} \leqq 1 + 4\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$$

$$2\left(\sin\frac{\gamma}{2}\sin\frac{\alpha}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right) \leqq \sin\frac{\alpha}{2} + 4\prod \sin\frac{\alpha}{2}.$$

To this inequality we add the inequality

$$2\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = \cos\frac{\beta-\gamma}{2} - \sin\frac{\alpha}{2} \leqq 1 - \sin\frac{\alpha}{2},$$

use formula (E), and have (29).

25. Multiplying (b) and (h) we get:

(k) $$\prod \sin \alpha \leqq \frac{3\sqrt{3}}{8}.$$

By formula (J), the arithmetic-geometric mean inequality and (k) we get

$$\frac{2}{3} \sum \cot \alpha = \frac{1}{3} \sum \frac{\sin \alpha}{\sin \beta \sin \gamma} \geqq \frac{1}{\sqrt[3]{\Pi \sin \alpha}} \geqq \frac{2}{\sqrt{3}},$$

(l) $$\sum \cot \alpha \geqq \sqrt{3}.$$

Using σ on (l) we get:

(m) $$\sum \tan \frac{\alpha}{2} \geqq \sqrt{3}.$$

Finally, multiplying (m) by $\prod \cot \frac{\alpha}{2}$ we get:

(30) $$\sum \cot \frac{\beta}{2} \cot \frac{\gamma}{2} \geqq \sqrt{3} \prod \cot \frac{\alpha}{2}.$$

26. We shall prove the inequality

(31) $$3 \sum \cot \alpha \geqq \prod \cot \frac{\alpha}{2}.$$

Proof. By the arithmetic-square mean inequality we have

$$\sum \sin^2 \alpha \geqq \sum \sin \beta \sin \gamma.$$

Using $\sin \beta \sin \gamma = \cos \beta \cos \gamma + \cos \alpha$ and formula (H) we restate this inequality as

$$2\left(1 + \prod \cos \alpha\right) \geqq \sum \cos \alpha + \sum \cos \beta \cos \gamma.$$

Adding $1 + \prod \cos \alpha$ to both sides we get

$$3\left(1 + \prod \cos \alpha\right) \geqq \prod (1 + \cos \alpha).$$

We divide this by $\prod \sin \alpha$, use formula (K), and have (31).

27. Combining (22) and (31) we get:

$$\sum \csc \alpha \leqq 2 \sum \cot \alpha,$$

from which we get by σ:

(32) $$\sum \sec \frac{\alpha}{2} \leqq 2 \sum \tan \frac{\alpha}{2}.$$

28. For the next three inequalities I know no strictly elementary proofs. My proof of the first inequality

(33) $$\sum \cos \beta \cos \gamma \leqq \frac{2}{\sqrt{3}} \prod \sin \alpha$$

will depend on the inequality

(n) $$s^2 \leqq 4R^2 + \frac{11}{3\sqrt{3}} F$$

stated (correctly) in 7.4, but misstated in 7.3. For a proof the reader may consult the two papers indicated in 7.3 and 7.4. NAKAJIMA's proof is not quite elementary as it uses calculus. As a preparation for the proof we translate (33) by means of formulas from 10. The translated inequality reads:

(o) $$r^2+s^2 \leqq 4R^2+\frac{4}{\sqrt{3}}rs.$$

In view of (n), to prove (o) we need only show that

$$r^2+\frac{11}{3\sqrt{3}}rs \leqq \frac{4}{\sqrt{3}}rs.$$

This inequality we easily reduce to

$$s \geqq 3\sqrt{3}\,r,$$

a well-known fact (see 5.11), which we later shall derive from our earlier results (see 36).

Using σ on (33) we get:

(34) $$\sum \sin\frac{\beta}{2}\sin\frac{\gamma}{2} \leqq \frac{2}{\sqrt{3}}\prod \cos\frac{\alpha}{2}.$$

Dividing (34) by $\prod \sin\frac{\alpha}{2}$ we get:

(35) $$\sum \csc\frac{\alpha}{2} \leqq \frac{2}{\sqrt{3}}\prod \cot\frac{\alpha}{2}.$$

29. We state the three conjectures (see 1):

(Cj_1) $$\sum \cos\beta\cos\gamma \geqq \frac{9\sqrt{3}}{4}\prod \cot\alpha,$$

(Cj_2) $$\sum \sin\frac{\beta}{2}\sin\frac{\gamma}{2} \geqq \frac{9\sqrt{3}}{4}\prod \tan\frac{\alpha}{2},$$

(Cj_3) $$\sum \csc\frac{\alpha}{2} \geqq \frac{9\sqrt{3}}{4}\prod \sec\frac{\alpha}{2}.$$

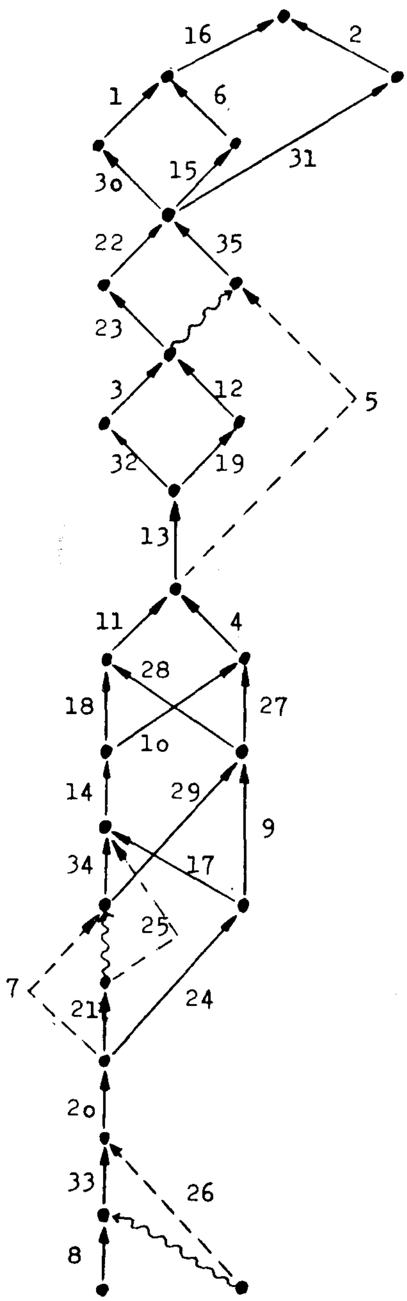
In essence there is only one conjecture: Clearly (Cj_2) and (Cj_3) are equivalent, i.e. easily derived from each other, and (Cj_2) follows from (Cj_1) by σ. Similarly, in the acute case (Cj_1) follows from (Cj_2) by τ. If there is a simple proof of (Cj_1) in the obtuse case — I do not know whether there is — it is sufficient to prove any one of the three conjectures. Anyway it will suffice to prove (Cj_1).

30. We shall now describe the graph, hinted at in 1 and shown on page 19. For clearness's sake we shall first assume that all conjectures are true and neglect the broken arrows in the graph (but retain the undulating ones!).

The 26 vertices of the graph represent the 26 normalized functions (see 5); which one represents which, will be clear later. An arrow means: "$\leqq$, with equality if and only if $\alpha=\beta=\gamma=\frac{\pi}{3}$". The undulating arrows represent the conjectures. The 35 proved fundamental inequalities are, in the graph, indicated by their numbers. The reader will now be able to find out, not only the identity of each vertex but also of each of the conjectures.

It is recommended that the reader makes his own drawing, on which he writes at each vertex its corresponding expression; some functions have two or even three different expressions; it pays to write them all.

We have described the graph under the assumption that the conjectures are facts. The broken arrows represent inequalities aimed at making life without the conjectures a bit easier.



The graph.

31. Assume again that the conjectures are facts. Then the graph (without the broken arrows) is complete in the following sense: Every generally valid inequality between normalized functions (see details in 5) is represented in the graph by an arrow or a directed string of arrows (transitivity!). To prove this it is sufficient to show that the functions in each of the following pairs cannot be the two sides of a generally valid inequality:

$$\left(\frac{1}{8}\prod \csc\frac{\alpha}{2},\ \frac{\sqrt{3}}{3}\sum \cot\alpha\right),$$

$$\left(\frac{1}{9}\sum \cot\frac{\beta}{2}\cot\frac{\gamma}{2},\ \frac{1}{12}\sum \csc\frac{\beta}{2}\csc\frac{\gamma}{2}\right)$$

$$\left(\frac{1}{9}\sum \cot\frac{\beta}{2}\cot\frac{\gamma}{2},\ \frac{\sqrt{3}}{3}\sum \cot\alpha\right),$$

$$\left(\frac{1}{12}\sum \csc\frac{\beta}{2}\csc\frac{\gamma}{2},\ \frac{\sqrt{3}}{3}\sum \cot\alpha\right)$$

$$\left(\frac{\sqrt{3}}{6}\sum \csc\alpha,\ \frac{1}{6}\sum \csc\frac{\alpha}{2}\right),$$

$$\left(\frac{\sqrt{3}}{3}\sum \tan\frac{\alpha}{2},\ \frac{1}{4}\sum \sec\frac{\beta}{2}\sec\frac{\gamma}{2}\right)$$

$$\left(\frac{2\sqrt{3}}{9}\sum \cos\frac{\alpha}{2},\ \frac{2}{3}\sum \sin\frac{\alpha}{2}\right),$$

$$\left(\frac{4}{9}\sum \cos\frac{\beta}{2}\cos\frac{\gamma}{2},\ \frac{2}{3}\sum \cos\alpha\right)$$

$$\left(\frac{2\sqrt{3}}{9}\sum \sin\alpha,\ \frac{2}{3}\sum \cos\alpha\right),$$

$$\left(\frac{4}{3}\sum \sin\frac{\beta}{2}\sin\frac{\gamma}{2},\ \frac{4}{9}\sum \sin\beta\sin\gamma\right)$$

$$\left(3\sqrt{3}\prod \tan\frac{\alpha}{2},\ \frac{4}{9}\sum \sin\beta\sin\gamma\right),$$

$$\left(8\prod \cos\alpha,\ 3\sqrt{3}\prod \cot\alpha\right).$$

Two of these cases are covered by the theorems (i) and (j) in 22, and the other cases can be covered by choosing suitable examples. We shall leave this tedious work to the reader, only advicing him that the two limiting cases $(0, 0, \pi)$ and $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$ of (α, β, γ) can be useful.

Should the conjectures be false a certain amount of work remains in order to complete the graph; clearly the broken arrows are not sufficient.

Some applications of the graph.

32. From now on we stop indicating fundamental inequalities by their numbers. If the reader has drawn the graph as recommended in 30, he will rapidly locate each inequality from the graph stated in the following.

Of the more than 300 inequalities readable from the graph only a few are found in [GI], namely in the following sections: 2.1, 2.7, 2.9, 2,12, 2.15, 2.16, 2.22, 2.23, 2.27, 2.28, 2.33, 2.34, 2.38, 2.40, 2.41, 2.42, 2.49, 2.51, 2.53, 2.54, 2.55 (the first inequality is valid only in acute triangles), 2.56 (see 24) and 2.63; among those are most of our lemmas.

33. One of the more interesting non-fundamental inequalities in the graph is

$$\prod \tan \frac{\alpha}{2} \geqq \prod \cot \alpha.$$

Multiplying by $\prod \sin \alpha$ we get:

$$\prod (1-\cos \alpha) \geqq \prod \cos \alpha .$$

From this we get by σ:

$$\prod \left(1-\sin \frac{\alpha}{2}\right) \geqq \prod \sin \frac{\alpha}{2} .$$

34. The second inequality in 33 may be restated as

$$\sum \cos \alpha \leqq 1-2 \prod \cos \alpha + \sum \cos \beta \cos \gamma .$$

By means of formula (G) this can be rewritten as

$$\sum \cos \alpha \leqq \sum \cos^2 \alpha + \sum \cos \beta \cos \gamma .$$

Adding $\sum \cos \beta \cos \gamma$ to both sides we get:

$$\sum \sin\beta \sin \gamma \leqq \left(\sum \cos \alpha\right)^2.$$

The author regards the proof above as an answer to the request by the editor at the end of [LC].

35. As $3 \sum \cot \alpha \geqq \sum \cot \frac{\alpha}{2}$ formula (O) yields

$$\sum \cot^2 \frac{\alpha}{2} \geqq \left(\sum \cot \frac{\alpha}{2}\right)\left(\sum \cot \alpha\right).$$

Compare 2.44.

We replace the left hand side by $\left(\sum \cot \frac{\alpha}{2}\right)^2 - 2 \sum \cot \frac{\beta}{2} \cot \frac{\gamma}{2}$, divide by $\sum \cot \frac{\alpha}{2} = \prod \cot \frac{\alpha}{2}$, and have the inequality

$$\sum \cot \frac{\alpha}{2} \geqq \sum \cot \alpha + 2 \sum \tan \frac{\alpha}{2}.$$

As $\cot \frac{x}{2} - \tan \frac{x}{2} = 2 \cot x$ the last inequality above can also be deduced from the inequality $3 \sum \tan \frac{\alpha}{2} \leqq \sum \cot \frac{\alpha}{2}$.

36. Every inequality in the graph, not containing any of the expressions

$$\sum \sin \frac{\alpha}{2}, \ \sum \sin \frac{\beta}{2} \sin \frac{\gamma}{2}, \ \sum \cos \frac{\alpha}{2}, \ \sum \cos \frac{\beta}{2} \cos \frac{\gamma}{2}, \ \sum \csc \frac{\alpha}{2},$$

$$\sum \csc \frac{\beta}{2} \csc \frac{\gamma}{2}, \ \sum \sec \frac{\alpha}{2}, \ \sum \sec \frac{\beta}{2} \sec \frac{\gamma}{2},$$

may be translated, by means of formulas from 10, into inequalities between functions of s, R, r.

In this manner the inequality $\prod \cos \frac{\alpha}{2} \leqq \frac{3\sqrt{3}}{8}$ yields

$$s \leqq \frac{3\sqrt{3}}{2} R,$$

compare 5.3. The inequality $\sum \tan \frac{\alpha}{2} \geqq \sqrt{3}$ yields the stronger inequality

$$s\sqrt{3} \leqq 4R + r.$$

Compare 5.5, 5.25, 5.33 (see the proof) and 7.2. The inequality

$$\prod \sin \alpha \geqq \frac{27}{8} \prod \cot \alpha$$

yields the inequality

$$s \leqq \frac{R(2R+r)}{\sqrt{R^2 - \frac{8}{27} r^2}}.$$

This inequality is stronger than both 5.7 and the right hand inequality of 5.8, i.e.

$$s^2 \leqq 4R^2 + 4Rr + 3r^2.$$

All other estimations above of s by functions of R, r obtained in the manner indicated are weaker than the last mentioned inequality, which may be derived by translating the inequality proved in 34.

Turning now to estimations below of s in terms of R, r, we first note that $\prod \cot \frac{\alpha}{2} \geqq 3\sqrt{3}$ yields

$$s \geqq 3\sqrt{3}\, r.$$

Compare 4.2 and 5.11. $\prod \cos \frac{\alpha}{2} \geqq \frac{27}{8} \prod \tan \frac{\alpha}{2}$ yields the stronger one:

$$2s^2 \geqq 27\, Rr,$$

compare 5.12. The inequality $3 \sum \cot \alpha \geqq \prod \cot \frac{\alpha}{2}$ yields

$$s^2 \geqq 3r(4R+r),$$

which is the strongest estimation below obtainable in the manner considered here. Compare 5.5 and 5.6.

I should very much like to see a derivation from the inequalities in the graph of the left hand inequality in 5.8, which is the strongest estimation below of s^2 by a homogeneous polynomial (of the second degree) in R, r.

37. Let β denote the middle angle. In view of the formulas $\sum \sin \alpha = \frac{s}{R}$ and $\sum \cos \alpha = \frac{R+r}{R}$ theorem (i) of 22 yields

$$\beta < \frac{\pi}{3} \Rightarrow s < \sqrt{3}\,(R+r)$$

$$\beta = \frac{\pi}{3} \Rightarrow s = \sqrt{3}\,(R+r)$$

$$\beta > \frac{\pi}{3} \Rightarrow s > \sqrt{3}\,(R+r).$$

In particular, s and $(R+r)\sqrt{3}$ are incomparable in the sense that no generally valid inequality between them exists.

38. Those inequalities from the graph, which contain just one of the expressions mentioned in the beginning of 36, are capable of partial translation by formulas from 10. From $\sum \cos \frac{\alpha}{2} \geqq \sqrt{3} \sum \cos \alpha$ and $\sum \sin \frac{\alpha}{2} \geqq \sum \cos \alpha$ we get:

$$\sum \cos \frac{\alpha}{2} \geqq \frac{\sqrt{3}(R+r)}{R}$$

$$\sum \sin \frac{\alpha}{2} \geqq \frac{R+r}{R}.$$

From the inequalities

$$\sum \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \geqq \frac{\sqrt{3}}{2} \sum \sin \alpha \quad \text{and} \quad \sum \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \leqq \frac{1}{2} \sum \cos \alpha$$

we get by subtraction and partial translation the inequality

$$\sum \sin \frac{\alpha}{2} \geqq \frac{s\sqrt{3}-(R+r)}{2R}.$$

The two estimations of $\sum \sin \frac{\alpha}{2}$ are independent of each other in the sense that, depending on the triangle, one or the other may be the stronger one.

From 37 it easily follows that $\sum \sin \frac{\alpha}{2} \geqq \frac{R+r}{R}$ is the stronger one (the weaker one) if $\beta < \frac{\pi}{3}$ $\left(\text{if } \beta > \frac{\pi}{3}\right)$, and that they coincide if $\beta = \frac{\pi}{3}$, β being the middle angle.

We also note that $\sum \sin \frac{\alpha}{2} \geqq \frac{1}{\sqrt{3}} \sum \sin \alpha$ yields the estimation

$$\sum \sin \frac{\alpha}{2} \geqq \frac{s}{R\sqrt{3}},$$

although this estimation can never be better than both the preceding ones because $s/(R\sqrt{3})$ is a weighted mean (with weights 1 and 2) of the two preceding right members.

39. As $\sin \frac{\alpha}{2} = \frac{r}{AI}$ etc., we conclude from 38 the following three inequalities:

$$\frac{1}{AI} + \frac{1}{BI} + \frac{1}{CI} \geqq \frac{1}{r} + \frac{1}{R},$$

$$\frac{1}{AI} + \frac{1}{BI} + \frac{1}{CI} \geqq \frac{s\sqrt{3} - (R+r)}{2Rr},$$

$$\frac{1}{AI} + \frac{1}{BI} + \frac{1}{CI} \geqq \frac{s}{rR\sqrt{3}}.$$

From $\sum \sin \frac{\alpha}{2} \leqq \frac{3}{2}$ we get an estimation above:

$$\frac{1}{AI} + \frac{1}{BI} + \frac{1}{CI} \leqq \frac{3}{2r}.$$

Observing that $r^3/(\prod AI) = \prod \sin \frac{\alpha}{2} = r/(4R)$ we have

$$AI \cdot BI \cdot CI = 4Rr^2,$$

Multiplying the four inequalities above by $4Rr^2$ we get the following four estimations of $\sum BI \cdot CI$:

$$BI \cdot CI + CI \cdot AI + AI \cdot BI \geqq 4r(R+r),$$

$$BI \cdot CI + CI \cdot AI + AI \cdot BI \geqq 2F\sqrt{3} - 2r(R+r),$$

$$BI \cdot CI + CI \cdot AI + AI \cdot BI \geqq \frac{4}{\sqrt{3}} F,$$

$$BI \cdot CI + CI \cdot AI + AI \cdot BI \leqq 6Rr.$$

The chain $6 \leqq \sum \csc \frac{\alpha}{2} \leqq \frac{2}{\sqrt{3}} \prod \cot \frac{\alpha}{2}$ yields the chain

$$6r \leqq AI + BI + CI \leqq \frac{2s}{\sqrt{3}},$$

which at right is stronger than 12.1, and as above the chain

$$\frac{3}{2Rr} \leqq \frac{1}{BI \cdot CI} + \frac{1}{CI \cdot AI} + \frac{1}{AI \cdot BI} \leqq \frac{s}{2\sqrt{3}\,Rr^2} \leqq \frac{3}{4r^2}$$

(the last inequality follows from $s \leqq \frac{3\sqrt{3}}{2} R$).

Finally, $2 \sum \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \leqq \sum \cos \alpha$ yields the two inequalities

$$\frac{1}{BI \cdot CI} + \frac{1}{CI \cdot AI} + \frac{1}{AI \cdot BI} \leqq \frac{R+r}{2Rr^2},$$

$$AI + BI + CI \leqq 2(R+r),$$

compare 12.2.

40. As $\csc x - \cot x = \frac{1-\cos x}{\sin x} = \tan \frac{x}{2}$ the inequality $\sqrt{3} \leqq \sum \tan \frac{\alpha}{2}$ yields:

$$\sum \csc \alpha \geqq \sum \cot \alpha + \sqrt{3}.$$

Using σ we get:

$$\sum \sec \frac{\alpha}{2} \geqq \sum \tan \frac{\alpha}{2} + \sqrt{3}.$$

Putting $\sum \tan \frac{\alpha}{2} = \frac{4R+r}{s}$ we get:

$$\frac{4R+r}{s} \leqq \sum \sec \frac{\alpha}{2} - \sqrt{3},$$

from which one can deduce the right hand inequality in 4.8.

41. By adding the inequalities

$$\sum \csc \frac{\alpha}{2} \leqq \frac{2}{3} \sum \cot \frac{\beta}{2} \cot \frac{\gamma}{2} \text{ and } 3 \leqq \frac{1}{3} \sum \cot \frac{\beta}{2} \cot \frac{\gamma}{2}$$

we get:

$$3 + \sum \csc \frac{\alpha}{2} \leqq \sum \cot \frac{\beta}{2} \cot \frac{\gamma}{2}.$$

Using τ on this we get 2.63, which is only valid in the acute case.

42. By adding the inequalities

$$\frac{\sqrt{3}}{2} \sum \cos \alpha \leqq \frac{1}{2} \sum \cos \frac{\alpha}{2} \text{ and } \frac{1}{2} \sum \sin \alpha \leqq \frac{\sqrt{3}}{2} \sum \sin \frac{\alpha}{2}$$

we get:

$$\sum \cos \left(\alpha - \frac{\pi}{6}\right) \leqq \sum \cos \left(\frac{\alpha}{2} - \frac{\pi}{3}\right).$$

43. Dividing the inequality $s\sqrt{3} \leqq 4R + r$ (see 36) by R and using that

$$\sum \sin \alpha = \frac{s}{R} \text{ and } \sum \cos \alpha = \frac{R+r}{R}$$

we get:

$$\sqrt{3}\sum \sin\alpha \leqq 3+\sum \cos\alpha .$$

By σ we get:

$$\sqrt{3}\sum \cos\frac{\alpha}{2} \leqq 3+\sum \sin\frac{\alpha}{2} .$$

The two inequalities can be rewritten as follows:

$$\sum \cos\left(\alpha+\frac{\pi}{3}\right) \geqq -\frac{3}{2}$$

$$\sum \cos\left(\frac{\alpha}{2}+\frac{\pi}{6}\right) \leqq \frac{3}{2} .$$

The last one can, alternatively, be proved by replacing (α, β, γ) with $\left(\frac{\alpha}{2}+\frac{\pi}{6}, \frac{\beta}{2}+\frac{\pi}{6}, \frac{\gamma}{2}+\frac{\pi}{6}\right)$ in the inequality $\sum \cos\alpha \leqq \frac{3}{2}$.

44. Many non-goniometric inequalities in [GI] can directly be translated into goniometric inequalities from our graph. For example can 6.11 be translated into $2\sum \sin\beta \sin\gamma \leqq 3\sum \cos\alpha$, one of our fundamental inequalities.

We shall conclude with another example which is more instructive. After division by $2F=4R^2 \prod \sin\alpha$, 6.4 can be translated into the inequality $3\sum \csc\alpha \geqq 4\sum \sin\alpha$, a very weak inequality from our graph. We extend it to the chain

$$\sum \csc\alpha \geqq \frac{9}{4}\prod \sec\frac{\alpha}{2} \geqq 2\sum \tan\frac{\alpha}{2} \geqq 2\sqrt{3} \geqq \frac{4}{3}\sum \sin\alpha .$$

Translating this chain and multiplying by $2F=2rs$ we get:

$$bc+ca+ab \geqq 18\,Rr \geqq 4r(4R+r) \geqq 4F\sqrt{3} \geqq \frac{4}{3}\,(h_b h_c + h_c h_a + h_a h_b) .$$

Compare 4.5, 5.16 (2), 6.18 and 7.2.

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