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334. SOME ADDITIONS TO KAMKE'S TREATISE*

A. M. Eĭšinskiĭ and *J. D. Kečkić*

1. We give a list of second order linear differential equations with their general solutions in closed form. None of these equations appears in KAMKE's book [1].

In what follows C_1 and C_2 denote arbitrary constants.

1° $x^4y''+xy'+y=0.$

$$y=\left(x-\frac{1}{x}\right)\left(C_1+C_2\int\frac{\exp(1/2x^2)}{1-2x+x^2}dx\right).$$

2° $x^4y''+(x^3+x^2)y'-(x^2-x+2)y=0.$

$$y=C_1xe^{\frac{2}{x}}+C_2(x+3)e^{-\frac{1}{x}}.$$

3° $x^4y''-xy'+y=0.$

$$y=\left(\frac{1}{x}-x\right)\left(C_1+C_2\int\frac{\exp(1/2x^2)}{x^2+2x+1}dx\right).$$

4° $x^4y''-(x^3+x)y'+(x^3+3)y=0.$

$$y=x\exp(-1/2x^2)\left(C_1+C_2\int\frac{\exp(1/2x^2)}{x}dx\right).$$

5° $x^4y''-(x^2-x^3)y'-(x^2+x+2)y=0.$

$$y=C_1xe^{-\frac{2}{x}}+C_2(x-3)e^{\frac{1}{x}}.$$

6° $4x^4y''-4x^3y'+(4x^2-2x-1)y=0.$

$$y=xe^{\frac{1}{2x}}\left(C_1+C_2\int\frac{\exp(-1/x)}{x}dx\right).$$

7° $x^4y''+(1-2x^2)y=0.$

$$y=C_1x\sin\left(\frac{1+C_2x}{x}\right)+C_2x\cos\left(\frac{1+C_2x}{x}\right).$$

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8° $4x^4y''-4x^3y'+(4x^2+2x-1)y=0.$

$$y=x\exp(-1/2x)\Big(C_1+C_2\int\frac{\exp(1/x)}{x}dx\Big).$$

9° $x^4y''-(a+2x^2)y=0.$

$$y=C_1(\sqrt{a}x-x^2)e^{\frac{\sqrt{a}}{x}}+C_2(\sqrt{a}x+x^2)e^{-\frac{\sqrt{a}}{x}}.$$

10° $x^4y''+(a^2-6x^2)y=0$, with $a\neq 0$.

$$y=C_1\Big[\frac{3}{a}x^2\cos\Big(\frac{a}{x}+C_2\Big)+\Big(1-\frac{3x^2}{a^2}\Big)\sin\Big(\frac{a}{x}+C_2\Big)\Big].$$

11° $x^5y''+x^2y'+(1-2x)y=0.$

$$y=xe^{\frac{1}{x}}\left(C_1+C_2\int\frac{\exp\Big(\frac{1}{2x^2}-\frac{2}{x}\Big)}{x^2}dx\right).$$

12° $x^5y''+x^4y'+(2x^3+x^2-4x-1)y=0.$

$$y=x\exp(1/2x^2)\Big(C_1+C_2\int\exp\Big(-\frac{1+2x}{2x^2}\Big)dx\Big).$$

13° $x^5y''-x^2y'+(2x+1)y=0.$

$$y=xe^{-\frac{1}{x}}\left(C_1+C_2\int\frac{\exp\Big(\frac{1}{2x^2}+\frac{2}{x}\Big)}{x^2}dx\right).$$

14° $x^5y''+(-2x^4+x^3+x^2)y'+(2x^3-x^2-4x-1)y=0.$

$$y=x\exp(1/2x^2)\Big(C_1+C_2\int\exp\Big(\frac{-1+2x}{2x^2}\Big)dx\Big).$$

15° $(-3x^5+x^4)y''+(4x^2-9x^3)y'+(9x^2-10x+3)y=0.$

$$y=xe^{\frac{1}{x}}\Big(C_1+C_2\int\frac{\exp(2/x)\,(1-3x)^2}{x^4}dx\Big).$$

16° $(x^5-2x^4)y''+(4x^3-3x^2)y'+(-4x^2+6x-1)y=0.$

$$y=xe^{\frac{1}{x}}\Big(C_1+C_2\int\frac{x^{-3/5}\exp(-1/2x)}{(2-x)\sqrt[4]{2-x}}dx\Big).$$

17° $x^6y''-4x^3y'+(6x^2+4)y=0.$

$$y=\exp(-1/x^2)\ (C_1x+C_2).$$

18° $x^6y''+4x^3y'+(4-6x^2)y=0.$

$$y=\exp(1/x^2)\ (C_1x+C_2).$$

19° $x^6y''-2ax^3y'+(2ax^2+a^2)y=0.$

$$y=x\exp(a/2x^2)\begin{cases}C_1\operatorname{ch}(\sqrt{a}/x)+C_2\operatorname{sh}(\sqrt{a}/x) & \text{for } a>0,\\ C_1\cos(\sqrt{-a}/x)+C_2\sin(\sqrt{-a}/x) & \text{for } a<0.\end{cases}$$

20° $x^6y''+x^5y'-(x^4+a)y=0.$

$$y=\begin{cases}C_1x\cos(\sqrt{-a}/2x^2)+C_2x\sin(\sqrt{-a}/2x^2) & \text{for } a<0,\\ C_1x\operatorname{ch}(\sqrt{-a}/2x^2)+C_2x\operatorname{sh}(\sqrt{-a}/2x^2) & \text{for } a>0.\end{cases}$$

21° $x^6y''+x^2y'-(x^2+3x+1)y=0.$

$$y=x\exp\left(\frac{1+3x^2}{3x^3}\right)\left(C_1+C_2\int\frac{1}{x^2}\exp\left(-\frac{1+6x^2}{3x^2}\right)dx\right).$$

22° $4x^6y''-(8x^5+4x^3)y'+(8x^4+4x^2+1)y=0.$

$$y=x^2\exp(-1/4x^2)\left(C_1\exp\frac{1}{x}\sqrt{\frac{3}{2}}+C_2\exp-\frac{1}{x}\sqrt{\frac{3}{2}}\right).$$

23° $x^9y''+(x^4-x^5)y'+(3x^4-x^3-1)y=0.$

$$y=x\exp(-1/3x^3)\left(C_1+C_2\int\frac{1}{x^2}\exp\left(\frac{3+4x}{12x^4}\right)dx\right).$$

24° $x^9y''+(x^5+x^4)y'+(1-x^3-3x^4)y=0.$

$$y=x\exp(1/3x^3)\left(C_1+C_2\int\frac{1}{x^2}\exp\left(\frac{3-4x}{12x^4}\right)dx\right).$$

2. Since the general solution of the differential equation

(1) $$f^2Fy''-(2ff'F+f^2F')y'+(2f'^2F+ff'F'-ff''F)y=0$$

is given by

(2) $$y=f(x)\left(C_1+C_2\int F(x)\,dx\right)$$

where C_1, C_2 are arbitrary constants, then starting with the solution (2) and taking particular forms for f and F, one can always construct an equation of the form (1) so that (2) is its general solution.

(Naturally, one cannot write an arbitrary linear differential equation of second order

$$Ay''+By'+Cy=0$$

in the form (1), since one first has to solve the same differential equation

$$Af''+Bf'+Cf=0,$$

in order to obtain f which appears in (1).)

However, taking

$$f(x)=P(x)\exp(Q(x)),\qquad F(x)=S(x)\exp(R(x)),$$

where P, Q, R, S are rational functions, the coefficients of equation (1) will always be polynomials in x. This is the case with the above equations. The only exception is equation 16, but again it is not difficult to see that sometimes if P, Q, R, S are algebraic functions one can also arrive at an equation whose coefficients are polynomials.

For example, let

$$P(x)=x^p,\quad Q(x)=kx^q,\quad R(x)=\frac{ax+b}{cx^2+dx+f}\quad\text{and}\quad S(x)=\frac{1}{Ax^2+Bx+C}.$$

Equation (1) then becomes

$$\begin{aligned}
&x^{2p}(Ax^2+Bx+C)(cx^2+dx+f)^2y''-\{2(px^{p-1}+kqx^{p+q-1})(Ax^2+Bx+C)\\
&\quad\times(cx^2+dx+f)^2+x^{2p}(Ax^2+Bx+C)[a(cx^2+dx+f)-(2cx+d)(ax+b)]\\
&\quad-x^{2p}(2Ax+B)(cx^2+dx+f)^2\}y'+\{2(px^{p-1}+kqx^{p+q-1})^2(Ax^2+Bx+C)\\
&\quad\times(cx^2+dx+f)^2+x^p(px^{p-1}+kqx^{p+q-1})(Ax^2+Bx+C)[a(cx^2+dx+f)\\
&\quad-(2cx+d)(ax+b)]-x^p(px^{p-1}+kqx^{p+q-1})(2Ax+B)(cx^2+dx+f)^2\\
&\quad-x^p[p(p-1)x^{p-2}+(2kpq+kq^2-kq)x^{p+q-2}+k^2q^2x^{p+2q-2}]\\
&\quad\times(Ax^2+Bx+C)(cx^2+dx+f)^2\}y=0,
\end{aligned}$$

and it contains equations 4, 6, 8, 11, 12, 13, 14, 17 and 18 from the above list, as well as equations 2.47, 2.50, 2.136, 2.181, 2.202, 2.347a, 2.349a, 2.351, 2.404 and 2.405 from [1].

In the same way it is possible to construct an equation which will contain all the above equations (and many more equations from [1]) as special cases. That equation and its solution would be too complicated to be of use. Indeed, even the above simple example contains 11 arbitrary constants.

Remark. We have heard later from A. M. Eĭšinskiĭ that A. M. Mikenberg and A. M. Eĭšinskiĭ [3] considered the following differential equation

$$s'ry''-r(rs''+2r's')y'+(-r''s'r+2r'^2s'+s''r'r)y+r^3s'^3\Phi\left(\frac{y}{r}\right)=0, \tag{3}$$

where s, r, Φ are sufficiently differentiable functions, stating that the substitution $y=r(x)K(s(x))$ reduces it to an integrable form.

Put $\Phi\left(\frac{y}{r}\right)\equiv 0$, $r=f$, $s'=F$. We obtain a differential equation wich differs from (1) only in the coefficient of y''. In order to examine why these equations differ in the first coefficient apply the transformation $y=r(x)K(s(x))$ to the differential equation

$$s'r^ay''-r(rs''+2r's')y'+(-r''s'r+2r'^2s'+s''r'r)y+r^3s'^3\,\Phi\left(\frac{y}{r}\right)=0,$$

to obtain

$$r^{a+1}K''+r^3\Phi(K)=0,$$

which shows that the only possible value of a is $a=2$. Therefore, owing to a misprint, equation (3) is not correct.

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Серова 3, кв. 39/7
Днепропетровск 70, СССР

Катедра за математику
Електротехнички факултет
Београд, Југославија