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323. CONVOLUTION INTEGRAL PROPERTY ON SOME CLASSES OF FUNCTIONS *

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1. Introduction

The very nature of the convolution integral in the domain of real variable t

$$c(t)=\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau \tag{1}$$

causes its known property that

$$c(t+\Delta t)\neq c(t)+\int_t^{t+\Delta t} f_1(\tau)f_2(t+\Delta t-\tau)\,d\tau$$

which appears as a serious disadvantage in most cases when numerical evalution of the convolution integral (1), for a number of values of its argument t, has to be carried out by means of numerical intergration. This stems from the fact that the integral in (1) does not lend itself to a common practice of saving the computational work through the sudbivision of the integration interval and using the integral value previously obtained over a part of the new integration interval.

This work will show the possibility that, for some classes of functions f in (1) a simple functional relationship, involving only elementary operations, can be established between $c(t)$ and $c(t+\Delta t)$, i. e. that a practically usable fuction $F[c(t), t, \Delta t]$ can be found such that

$$c(t+\Delta t)=F[c(t),t,\Delta t]+\int_t^{t+\Delta t} f_1(\tau)f_2(t+\Delta t-\tau)\,d\tau \tag{2}$$

which provides the possibility of overcoming the above mentioned inherent disadvantage of the convolution integral by substituting the repeated intergration over an interval $[0,t]$, by the evaluation of the function F from (2), each time the necessary convolution value $c(t)$, entering F, is already known by any means, anda new value $c(t+\Delta t)$ is cumputed for arbitrary $\Delta t>-t$.

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According to the well known relationship

$$\int_0^t f_1(\tau) f_2(t-\tau)\,d\tau = \int_0^t f_2(\tau) f_1(t-\tau)\,d\tau$$

existence of the function F, in the simple elementary form, is assured if any of two functions f_1 and f_2 belongs to the predetermined class of functions, with no more restrictions posed on the other one of them.

Practical interest for functions f from (2), which, to the extent of author's knowledge, have not been considered in the existing literature, arises from the fact that convolution integral-form functions very often happen to be the solutions to problems from the fields in which intensive scientific and engineering investigations are currently carried out, such as the fields of automatic regulation, electrical network processes, heat conduction etc. This interest is still more amplified by the fact that in quite a number of cases the considered integral cannot be calculated in a closed form, either because of the nature of fuctions f_1 or f_2, or because one of them is not given in analytical form.

2. The classes of functions

Let us arbitrarily assign the exponential function of the general form

$$f_e(t) = a^{bt} \tag{3}$$

to the first class of functions for which the corresponding function F in (2) exists. Substituting f_e for f_2 we get a special form of convolution integral (1)

$$c_e(t) = \int_0^t f_1(\tau) f_e(t-\tau)\,d\tau = \int_0^t f_1(\tau) a^{b(t-\tau)}\,d\tau$$

for which the function F from (2) is obtained as

$$F_e = \int_0^t f_1(\tau) a^{b(t+\Delta t-\tau)}\,d\tau = c_e(t)\cdot f_e(\Delta t) \tag{4}$$

Taking as the second class of functions sine and cosine functions of the form

$$f_s(t) = \sin\omega t \qquad f_c(t) = \cos\omega t \tag{5}$$

as the third class of functions of the hyperbolic sine and cosine functions of the form

$$f_{sh}(t) = \operatorname{sh}\omega t \qquad f_{ch}(t) = \operatorname{ch}\omega t \tag{6}$$

and, finally, as the fourth class of functions the power functions of the form

$$f_n(t) = t^n \tag{7}$$

and substituting them for f_2 we get the following special forms of the convolution integral (1) respectively

$$c_s(t)=\int_0^t f_1(\tau)\sin[\omega(t-\tau)]\,d\tau$$

$$c_c(t)=\int_0^t f_1(\tau)\cos[\omega(t-\tau)]\,d\tau$$

(8) $$c_{sh}(t)=\int_0^t f_1(\tau)\,\mathrm{sh}[\omega(t-\tau)]\,d\tau$$

$$c_{ch}(t)=\int_0^t f_1(\tau)\,\mathrm{ch}[\omega(t-\tau)]\,d\tau$$

$$c_n(t)=\int_0^t f_1(\tau)(t-\tau)^n\,d\tau$$

for which the corresponding functions F from (2) are simply derived in the manner already shown:

$$F_s=c_s(t)\cdot fc(\Delta t)+c_c(t)\cdot f_s(\Delta t)$$

$$F_c=c_c(t)\cdot f_c(\Delta t)-c_s(t)\cdot f_s(\Delta t)$$

(9) $$F_{sh}=c_{sh}(t)\cdot f_{ch}(\Delta t)+c_{ch}(t)\cdot f_{sh}(\Delta t)$$

$$F_{ch}={}_{ch}(t)\cdot f_{ch}(\Delta t)+c_{sh}(t)\cdot f_{sh}(\Delta t)$$

$$F_n=\sum_{i=0}^{n}\binom{n}{i}c_i(t)\,f_{(n-i)}(\Delta t).$$

3. Properties common to all four classes

When f_2 in (1) is an arbitrary sum of a finite number of functions (3), (5), (6) and (7), then the function F from (2) exists in the form

(10) $$F=\sum_{i=1}^{m}F_i$$

where all F_i are corresponding functions F from (4) and (9).

When a functions f_2 in (1) is formed by the p-oduct of a finite number of the same-class functions, then in the case of the first class, the function F retains the same general form (4); while for the second and third classes of functions products can be transformed in sums and resulting functions F obtained in the form of (10) afterwards.

Corresponding functions F from (2) can be found also for those functions f_2 in (1) consisting of products containing finite number of different-class functions, but they may have a more complex form.

It should be mentioned, at the end, that all the functions F from (2) must satisfy the following two identities:

$$F[c(t),\, t,\, 0] \equiv c(t)$$

$$F[c(t),\, 0,\, \Delta t] \equiv 0.$$

4. Conclusion

Existence of functions F from (2) in the elementary forms of (4), (9), (10) and other similar forms, has a consinderable practical significance for the numerical evalution of the convolution integral (1) for a number of values of its argument t. This signaficance stems from the fact that, according to the formula (2), the numerical integration over an interval $[0,t]$ can be substituted by the evaluation of the elementary function F, each time the necessary convolution value $c(t)$, entering F, is already known.

The extent to which this algorithm is more advantageous than the repeated numerical intergration over all intervals $[0,\, t_i]$, is generally dependent on classes of functions involved, number of argument values t_i, and number of subintervals used for obtaining the approximate value of the integral (1). As in the most situations number of argument values, for which the value of (1) has to be calculated, will considerably exceed two, the algorithm using functions F will inevitably prove its advantage in the great many of practical cases.

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