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311. A NOTE ON PATHS IN THE *p*-SUM OF GRAPHS*

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The number of the paths of length k in the p-sum of graphs is determined in this paper.

In this note we consider undirected graphs without loops or multiple edges.

In [1] the p-sum $(p=1, \ldots, n)$ of graphs $G_1, \ldots, G_n$ is defined. The set of vertices of the p-sum is the Cartesian product of the sets of vertices of graphs $G_1, \ldots, G_n$. If x_i and y_i are vertices of graphs $G_i\,(i=1, \ldots, n)$, the vertices of the p-sum $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are adjacent if, and only if, exactly p of n pairs (x_i, y_i) $(i=1, \ldots, n)$ are the pairs of the adjacent vertices in corresponding graphs and if for the other $n-p$ pairs holds $x_i=y_i$. If $p=n$ the p-sum is called the product of graphs and in the case $p=1$ the sum of graphs.

The adjacency matrix $A=\|a_{ij}\|_1^m$ of graph G with m vertices, is the matrix whose element a_{ij} is equal to 1 if the vertex i is adjacent to the vertex j and is equal to zero in the opposite case. The spectrum of the graph G is the set of solutions $\{\lambda_1, \ldots, \lambda_m\}$ of the characteristic equation $\det(A-\lambda I)=0$ of the matrix A, i.e. the set of the eigenvalues of A.

It was noticed in [2], that the adjacency matrix of the product $G_1\times G_2$ of graphs G_1 and G_2 is equal to the KRONECKER's product $A_1\otimes A_2$ of the adjacency matrices A_1 and A_2 of the graphs G_1 and G_2. In [3] the adjacency matrix of the sum G_1+G_2 was determined; it is of the form $A_1\otimes I_2+I_1\otimes A_2$, where I_1 and I_2 are the unit matrices of the same order as A_1 and A_2 respectively.

Let $A_1, \ldots, A_n$ be the adjacency matrices and $\{\lambda_{1s_1}\}, \ldots, \{\lambda_{ns_n}\}$ the spectrums of the graphs $G_1, \ldots, G_n$.

We have noticed in [4], that the adjacency matrix of the p-sum of graphs $G_1, \ldots, G_n$ is given by the expression

$$(1)\qquad \mathcal{A}=A_1\otimes\cdots\otimes A_p\otimes I_{p+1}\otimes\cdots\otimes I_n$$
$$+A_1\otimes\cdots\otimes A_{p-1}\otimes I_p\otimes A_{p+1}\otimes I_{p+2}\otimes\cdots\otimes \mathrm{I}_n+\cdots$$
$$+I_1\otimes\cdots\otimes I_{n-p}\otimes A_{n-p+1}\otimes\cdots\otimes A_n.$$

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In this note we determine the number of paths of the p-sum of graphs. The result is similar to the one which appears in Lemma in [5].

We denote with $\{X\}$ the sum of all the elements of the matrix X, with $\mathcal{N}_k$ the number of the paths of length k in the p-sum and with $N_k^1, \ldots, N_k^n$ the numbers of the paths of length k in the graphs $G_1, \ldots, G_n$.

It is known, that the number of paths of length k in a given graph can be determined by the use of the k-th power of the adjacency matrix of the graph (see, for example, [1] p. 127), so that the following formula holds:

$$\mathcal{N}_k = \{\mathcal{A}^k\}. \tag{2}$$

Matrix (1) represents the sum of $q = \binom{n}{p}$ summands (we denote them by $B_1, \ldots, B_q$), which are, according to [5], called the normal summands. With each normal summand B_r we associate the expression S_r. S_r contains as a factor the expression $\lambda_{i\,s_i}$ if and only if B_r contains A_i as a factor of the KRONECKER's product. S_r has no other factors.

In the same way as in [4] we have

$$\mathcal{A}^k = (B_1 + \cdots + B_q)^k = \sum_{j_1, \ldots, j_q} \frac{k!}{j_1! \cdots j_q!} B_1^{j_1} \cdots B_q^{j_q}$$
$$= \sum_{j_1, \ldots, j_q} \frac{k!}{j_1! \cdots j_q!} A_1^{l_1} \otimes \cdots \otimes A_n^{l_n} \quad (j_1 + \cdots + j_q = k), \tag{3}$$

where l_i is the sum of those numbers $j_1, \ldots, j_q$ which are in the expression $B_1^{j_1} \cdots B_q^{j_q}$ exponents of those normal summands which contain A_i $(i = 1, \ldots, n)$.

One may easily verify that $\{X \otimes Y\} = \{X\} \cdot \{Y\}$ holds and from (3) we obtain

$$\{\mathcal{A}^k\} = \sum_{j_1, \ldots, j_q} \frac{k!}{j_1! \cdots j_q!} \{A_1^{l_1}\} \cdots \{A_n^{l_n}\}$$

i.e.

$$\mathcal{N}_k = \sum_{j_1, \ldots, j_q} \frac{k!}{j_1! \cdots j_q!} N_{l_1}^1 \cdots N_{l_n}^n. \tag{4}$$

In the special cases $n = 2$, $p = 1$ (sum of graphs) and $n = 2$, $p = 2$ (product of graphs) we have

$$\mathcal{N}_k = \sum_{j=0}^{k} \binom{k}{j} N_j^1 N_{k-j}^2, \quad \mathcal{N}_k = N_k^1 N_k^2. \tag{5}$$

The number of paths of length k in a graph can always be representec in the form $N_k = \sum_s C_s \lambda_s^k$, where C_s are constants and λ_s are eigenvalues o the adjacency matrix A. Let $N_k^i = \sum_{s_i} C_{is_i} \lambda_{is_i}{}^k$, $(i = 1, \ldots, n)$ be the numbers of paths of length k in the graphs $G_1, \ldots, G_n$. Then (4) becomes

$$\mathcal{N}_k = \sum_{j_1, \ldots, j_q} \frac{k!}{j_1! \cdots j_q!} \sum_{s_1} C_{1s_1} \lambda_{1s_1}{}^{l_1} \cdots \sum_{s_n} C_{ns_n} \lambda_{ns_n}{}^{l_n}$$

$$= \sum_{s_1, \ldots, s_n} C_{1s_1} \cdots C_{ns_n} \sum_{j_1, \ldots, j_q} \frac{k!}{j_1! \cdots j_q!} \lambda_{1s_1}{}^{l_1} \cdots \lambda_{ns_n}{}^{l_n}$$

$$= \sum_{s_1, \ldots, s_n} C_{1s_1} \cdots C_{ns_n} \sum_{j_1, \ldots, j_q} \frac{k!}{j_1! \cdots j_q!} S_1^{j_1} \cdots S_q^{j_q}$$

$$= \sum_{s_1, \ldots, s_n} C_{1s_1} \cdots C_{ns_n} (S_1 + \cdots + S_q)^k$$

i.e.

(6) $$\mathcal{N}_k = \sum_{s_1, \ldots, s_n} C_{1s_1} \cdots C_{ns_n} (\lambda_{1s_1} \cdots \lambda_{ps_p} + \cdots + \lambda_{n-p+1\, s_{n-p+1}} \cdots \lambda_{ns_n})^k,$$

where the bracket contains the elementary symmetric function of n variables $\lambda_{1s_1}, \ldots, \lambda_{ns_n}$.

Some of these results can be easily extended to a larger class of graphs or to some operations on the graphs, which are not contained in the p-sum of graphs, but corresponding adjacency matrices can be expressed in terms of normal summands.

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