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309. AN INEQUALITY FOR CONVEX FUNCTIONS*

Petar M. Vasić and *Radovan R. Janić*

It is proved that under the conditions 1°—3°, for each functions f convex on $I'=[a, b']$, the inequality (1) holds.

Theorem. *If* $x_i \in [a, b] = I$ $(i=1, 2, \ldots, 2n+1)$ *and* p_i $(i=1, 2, \ldots, 2n+1)$ *are real numbers such that for every* $k=1, \ldots, n$:

1° $\quad p_1 > 0,\ p_{2k} \leqq 0,\ p_{2k}+p_{2k+1} \leqq 0,\ \sum_{i=1}^{2k} p_i \geqq 0,\ \sum_{i=1}^{2k+1} p_i > 0;$

2° $\quad x_{2k} \leqq x_{2k+1};$

3° $\quad \sum_{i=1}^{2k} p_i (x_i - x_{2k+1}) \geqq 0,$

then for any function f *convex on* $I'=[a, b']$, *where*

$$b' = \max_{1 \leqq k \leqq n} \left(b, \frac{\sum_{i=1}^{2k+1} p_i x_i}{\sum_{i=1}^{2k+1} p_i} \right)$$

the following inequality

$$\sum_{i=1}^{2n+1} p_i f(x_i) \leqq \left(\sum_{i=1}^{2n+1} p_i \right) f \left(\frac{\sum_{i=1}^{2n+1} p_i x_i}{\sum_{i=1}^{2n+1} p_i} \right) \tag{1}$$

is valid.

Inequality (1) is, in fact, the opposite inequality to JENSEN's inequality.

Remark. The formulation of the Theorem can be simplified if we consider the interval $[a, +\infty)$ instead of the interval $[a, b]$ because in this case we have $I = I'$.

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In order to prove this Theorem we shall use the following:

Lemma. If f is a convex function on I and if $a_1 \leqq a_2 \leqq a_3 \, (a_1, a_2, a_3 \in I)$, $c_1+c_2+c_3>0$, $c_1+c_2 \geqq 0$, $c_2+c_3 \geqq 0$, and $c_1 \geqq 0$, $c_3 \geqq 0$, then

$$(2) \qquad c_1 f(a_1)+c_2 f(a_2)+c_3 f(a_3) \geqq (c_1+c_2+c_3) f\left(\frac{c_1 a_1+c_2 a_2+c_3 a_3}{c_1+c_2+c_3}\right).$$

Proof of the Lemma. This is, in fact, the particular case (for $n=3$) of STEFFENSEN's inequality (see: [1], or [2] pp. 107—119).

Proof of the Theorem. We shall use mathematical induction. If we put

$$c_1=-p_2, \; c_2=-p_3, \; c_3=p_1+p_2+p_3,$$

$$a_1=x_2, \; a_2=x_3, \; a_3=\frac{p_1 x_1+p_2 x_2+p_3 x_3}{p_1+p_2+p_3},$$

where $p_1, p_2, p_3, x_1, x_2, x_3$, are real numbers such that

$$p_1>0, \; p_2 \leqq 0, \; p_2+p_3 \leqq 0, \; p_1+p_2 \geqq 0, \; p_1+p_2+p_3>0, \; x_2 \leqq x_3,$$

$$p_1(x_1-x_3)+p_2(x_2-x_3) \geqq 0,$$

the conditions for the application of the above Lemma are fulfilled. Then, the inequality (2) becomes

$$(3) \qquad p_1 f(x_1)+p_2 f(x_2)+p_3 f(x_3) \leqq (p_1+p_2+p_3) f\left(\frac{p_1 x_1+p_2 x_2+p_3 x_3}{p_1+p_2+p_3}\right).$$

This is the inequality (1) for $n=1$.

Suppose that this Theorem holds for any $n-1$ and suppose that the conditions of Theorem are fulfilled for n. Then the following inequality

$$(4) \qquad \sum_{i=1}^{2n-1} p_i f(x_i)+p_{2n} f(x_{2n})+p_{2n+1} f(x_{2n+1})$$

$$\leqq \left(\sum_{i=1}^{2n-1} p_i\right) f\left(\frac{\sum_{i=1}^{2n-1} p_i x_i}{\sum_{i=1}^{2n-1} p_i}\right)+p_{2n} f(x_{2n})+p_{2n+1} f(x_{2n+1})$$

also holds.

Putting

$$P_1=\sum_{i=1}^{2n-1} p_i, \; P_2=p_{2n}, \; P_3=p_{2n+1}, \; X_1=\frac{\sum_{i=1}^{2n-1} p_i x_i}{\sum_{i=1}^{2n-1} p_i}, \; X_2=x_{2n}, \; X_3=x_{2n+1},$$

then since, by hypothesis, the conditions 1°—3° are satisfied for n, we ha

$$\sum_{i=1}^{2n-1} p_i>0, \; p_{2n} \leqq 0, \; p_{2n}+p_{2n+1} \leqq 0, \; \sum_{i=1}^{2n} p_i \geqq 0, \; \sum_{i=1}^{2n+1} p_i>0, \; x_{2n} \leqq x_{2n+1},$$

$$\sum_{i=1}^{2n} p_i (x_i-x_{2n+1}) \geqq 0.$$

As the conditions for applying the inequality (3) (with P_k instead of p_k and X_k instead of x_k) are satisfied, the same may be applied on the right-hand side of the inequality (4), so that we obtain

$$\left(\sum_{i=1}^{2n-1} p_i\right) f\left(\frac{\sum_{i=1}^{2n-1} p_i x_i}{\sum_{i=1}^{2n-1} p_i}\right) + p_{2n} f(x_{2n}) + p_{2n+1} f(x_{2n+1}) \tag{5}$$

$$\leq \sum_{i=1}^{2n+1} p_i f\left(\frac{\sum_{i=1}^{2n+1} p_i x_i}{\sum_{i=1}^{2n+1} p_i}\right).$$

On the basis of (4) and (5) we conclude that the result is valid for n, if it holds for $n-1$, completing the induction and the proof of the theorem.

If $f(x) = x^2$, and

$$x_{2k} \leqq x_{2k-1}, \; x_{2k} \leqq x_{2k+1} \qquad (k = 1, \ldots, n) \tag{6}$$

and

$$p_i = (-1)^{i-1} \quad (i = 1, 2, \ldots, 2n+1),$$

the conditions for the applications of the Theorem are fulfilled, so that the inequality

$$\left(\sum_{k=1}^{2n+1} (-1)^{k-1} x_k\right)^2 \geqq \sum_{k=1}^{2n+1} (-1)^{k-1} x_k^2$$

holds.

This inequality was proved by Z. OPIAL (see [3], or [2] p. 351).

The hypotheses on x_i can be improved somewhat. Namely, instead of the assumptions (6) one can use the weaker assumptions

$$\sum_{j=1}^{k} (x_{2j-1} - x_{2j}) \geqq 0, \; x_{2k} \leqq x_{2k+1} \qquad (k = 1, \ldots, n). \tag{7}$$

Obviously (6) implies (7), and it is easy to verify that (7) implies 3°. This is a slight improvement of Z. OPIAL's result.

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Katedra za matematiku
Elektrotehnički fakultet,
Beograd, Jugoslavija