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305. EXTREME PROPERTIES OF PROPER VALUES OF UNITARY TRANSFORMATIONS*

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In a paper of A. Horn and R. Steinberg [2] a minimax principle concerning a unitary transformation on a unitary space has been stated without proof. In this article we supply a proof for a more general case of the proposition and give other generalizations.

1. Definitions and notations. We denote by E_n the unitary space of dimension n. The inner product of two vectors will be indicated by (ξ, η). If $\xi_1, \ldots, \xi_k$ are vectors, then $[\xi_1, \ldots, \xi_k]$ denotes the subspace spanned by them. If $\{\xi_1, \ldots, \xi_k\}$ is an orthonormal set of vectors in E_n, we write $\{\xi_i\}$ o.n. Two linear transformations A and B on E_n are said to be *congruent* if there exists a non-singular linear transformation X such that $B = X^*AX$.

If A is a linear transformation on E_n and if M is a subspace of E_n, we define a transformation $A \mid M$ on M as follows: if $\xi \in M$, we let $(A \mid M)\,\xi = PAP\xi$, where P is the orthogonal projection on M.

We observe that if ξ and η are in M, then

$$\big((A \mid M)\,\xi, \eta\big) = (PAP\xi, \eta) = (A\xi, \eta).$$

We shall use the symbol (a_{ij}) for the n-by-n matrix whose elements are a_{ij}, $i, j = 1, \ldots, n$. The determinant of (a_{ij}) will be denoted by $\det(a_{ij})$; also $\det A$ denotes the determinant of the linear transformation A.

2. Theorem. *Let A be a linear transformation on E_n. Then for any set of vectors $\{\xi_1, \ldots, \xi_n\}$ in E_n*

$$\det\big((A\xi_i, \xi_j)\big) = \det A \det\big((\xi_i, \xi_j)\big).$$

Proof. Let $\{\alpha_1, \ldots, \alpha_n\}$ be any orthonormal basis in E_n. Then we note that

$$(A\xi_i, \xi_j) = \sum_{k=1}^{n} \left\{ \sum_{l=1}^{n} [(\xi_i, \alpha_l)\,(\overline{\xi_j, \alpha_k})\,(A\alpha_l, \alpha_k)] \right\}.$$

This implies that

$$\det\big((A\xi_i, \xi_j)\big) = \det\left(\sum_{k=1}^{n} \left\{ \sum_{l=1}^{n} [(\xi_i, \alpha_l)\,(\overline{\xi_j, \alpha_k})\,(A\alpha_l, \alpha_k)] \right\}\right) = \det A \, \det\big((\xi_i, \xi_j)\big).$$

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3. Corollary. Let $\{\xi_i, \dots, \xi_k\}$ be linearly independent and $M=[\xi_1, \dots, \xi_k]$. Let A be the transformation in § 2. Then

$$\det\big((A\xi_i, \xi_j)\big)=\det(A\,|\,M)\det\big((\xi_i, \xi_j)\big).$$

Note that i and j run from 1 through k.

4. Theorem. *If A is congruent to a unitary transformation U with proper values $u_1, \dots, u_n$ and if $0<\arg u_1 \leqq \dots \leqq \arg u_n < \pi$ then $(A\xi, \xi)\neq 0$ for all $\xi\neq 0$ and*

$$\arg u_j = \inf_{\substack{\dim\\ S=j}} \sup_{\substack{\xi\in S\\ \xi\neq 0}} \arg (A\xi, \xi) = \sup_{\substack{\dim S\\ =n-j+1}} \inf_{\substack{\xi\in S\\ \xi\neq 0}} \arg (A\xi, \xi),$$

where S ranges over subspaces of E_n.

Note that U is a congruent to itself. Therefore the above lemma is satisfied by replacing A with U. This theorem is due to A. HORN and R. STEINBERG [2].

5. Theorem. *Let U be a unitary transformation on E_n with proper values $u_1, \dots, u_n$ such that $0<\arg u_1 \leqq \dots \leqq \arg u_n<\pi$. Let $1\leqq i_1<\dots<i_k\leqq n$ be a sequence of integers. Then*

$$\text{(1)}\qquad \arg u_{i_1}+\dots+\arg u_{i_k} = \inf_{\substack{M_1\subset\dots\subset M_k\\ \dim M_p=i_p}} \sup_{\substack{\xi_p\in M_p\\ \{\xi_p\}\ \text{o.n.}}} \arg\det\big((U\xi_i, \xi_j)\big).$$

Proof. Let $\{\gamma_1, \dots, \gamma_n\}$ be an orthonormal set such that $U\gamma_i=u_i\gamma_i$, $i=1, \dots, n$. Let $M_p=[\gamma_1, \dots, \gamma_{i_p}]$ and let $\{\xi_1, \dots, \xi_k\}$ be any orthonormal set with $\xi_p\in M_p$ for $p=1, \dots, k$. Let $M=[\xi_1, \dots, \xi_k]$ and $B=U\,|\,M$. We observe that the rank od B is k. Thus B on M is non-singular. Let $B=\sqrt{BB^*}\,V$ be the polar decomposition of B, where V is a unitary transformation on M. Then we observe that

$$\arg\det\big((U\xi_i, \xi_j)\big)=\arg\det B=\arg\det V.$$

Let $s_1, \dots, s_k$ be proper values of V. Thus

$$\arg s_1+\dots+\arg s_k=\arg\det\begin{bmatrix}(U\xi_1, \xi_1)\cdots(U\xi_1, \xi_k)\\ \vdots\\ (U\xi_k, \xi_1)\quad\ (U\xi_k, \xi_k)\end{bmatrix}.$$

Let δ_i be a proper vector of V such that $V\delta_i=s_i\delta_i$, $i=1, \dots, k$. Then

$$\big((U\,|\,M)\,\delta_i, \delta_i\big)=(\sqrt{BB^*}\,V\,\delta_i, \delta_i)=s_i(\sqrt{BB^*}\,\delta_i, \delta_i),\ i=1, \dots, k.$$

Therefore

$$\arg s_i=\arg\big((U\,|\,M)\,\delta_i, \delta_i\big)=\arg(U\delta_i, \delta_i).$$

By § 4 we have $\arg u_1\leqq\arg(U\delta_i, \delta_i)\leqq\arg u_n$. Thus

$$\arg u_1\leqq\arg s_i\leqq\arg u_n,$$

where $i=1, \dots, k$. So we can order $s_1, \dots, s_k$ in such a way that

$$0<\arg s_1\leqq\dots\leqq\arg s_k<\pi.$$

But

$$\arg s_p = \inf_{\substack{N_p \subset M \\ \dim N_p = p}} \sup_{\substack{\xi \in N_p \\ \xi \neq 0}} \arg (U\xi, \xi)$$

$$\leq \sup_{\substack{\xi \in [\xi_1, \dots, \xi_p] \\ \xi \neq 0}} \arg (U\xi, \xi) = \arg u_{i_p}, \quad p = 1, \dots, k.$$

Therefore

$$\arg u_{i_1} + \cdots + \arg u_{i_k} \geqq \arg s_1 + \cdots + \arg s_k.$$

Thus the left side of (1) is no smaller than its right side.

Now let M_j, $1 \leqq j \leqq k$, be subspaces such that $M_1 \subset \cdots \subset M_k$ and $\dim M_p = i_p$, $p = 1, \dots, k$. Let $N_p = [\gamma_{i_p}, \gamma_{i_p+1}, \dots, \gamma_n]$, $p = 1, \dots, k$. Then $\dim N_p = n - i_p + 1$ and $N_1 \supset \cdots \supset N_k$. By HORN's theorem [1, § 2.2] there exists a subspace M spanned by an orthonormal set $\{\xi_1, \dots, \xi_k\}$ where $\xi_p \in M_p$, and M is also spanned by another orthonormal set $\{\beta_1, \dots, \beta_k\}$ where $\beta_p \in N_p$, $p = 1, \dots, k$. Let $B = U | M$ and let $B = \sqrt{BB^*}\, V$ be the polar decomposition in B, where V is a unitary transformation on M. This implies that

$$\arg \det \big((U\beta_i, \beta_j)\big) = \arg \det \big((U\xi_i, \xi_j)\big) = \arg \det B = \arg \det V.$$

Let $b_1, \dots, b_k$ be proper values of V. Then

$$\arg \det \big((U\xi_i, \xi_j)\big) = \arg b_1 + \cdots + \arg b_k.$$

But by § 4 we have

$$\arg b_j = \sup_{\substack{N \subset M \\ \dim N \\ = j-1}} \inf_{\substack{\xi \perp N \\ \xi \in M \\ \xi \neq 0}} \arg(U\xi, \xi)$$

$$\geq \inf_{\substack{\xi \perp [\beta_1, \dots, \beta_{j-1}] \\ \xi \neq 0}} \arg (U\xi, \xi)$$

$$\geqq \inf_{\substack{\xi \in N_j \\ \xi \neq 0}} \arg (U\xi, \xi) = \arg u_{i_j}, \quad j = 1, \dots, k.$$

Therefore

$$\arg u_{i_j} + \cdots + \arg u_{i_j} \leqq \arg b_1 + \cdots + \arg b_k.$$

Thus the left side of (1) is no larger than its right side. Thus the proof is complete.

6. Theorem. *Let U be a unitary transformation on E_n and A be a linear transformation congruent to U. Let $u_1, \dots, u_n$ be proper values of U such that $0 < \arg u_1 \leqq \cdots \leqq \arg u_n < \pi$ and $1 \leqq i_1 < \cdots < i_k \leqq n$ a sequence of integers. Then*

$$\arg u_{i_1} + \cdots + \arg u_{i_k} = \inf_{\substack{M_1 \subset \cdots \subset M_k \\ \dim M_p = i_p}} \sup_{\xi_p \in M_p} (\arg a_1 + \cdots + \arg a_k),$$

where $\{\xi_1, \dots, \xi_k\}$ ranges over linearly independent sets of vectors and $a_1, \dots, a_k$ are proper values of the matrix $\big((A\xi_i, \xi_j)\big)$. Moreover the value $(\arg a_1 + \cdots + \arg a_k)$ *depends only on* $[\xi_1, \dots, \xi_k]$.

Proof. If A is congruent to U, then there exists a non-singular linear transformation X such $A = X^*UX$. This implies that

$$(X^*UX\xi_i, \xi_j) = (U\eta_i, \eta_j),$$

where, for example, $\eta_i = X\xi_i$. Therefore the set $\{\eta_1, \dots, \eta_k\}$is linearly independent, and we see that

$$\det\big((A\xi_i, \xi_j)\big) = \det U \det\big((\eta_i, \eta_j)\big).$$

Since the matrix $\big((\eta_i, \eta_j)\big)$ is positive

$$\arg \det\big((A\xi_i, \xi_j)\big) = \arg \det U.$$

Therefore by applying previous theorem the proof is complete.

7. Theorem. *Let U be a unitary transformation on E_n and A be a linear transformation congruent to U. Let $u_1, \dots, u_n$ be proper values of U such that $0 < \arg u_1 \leqq \dots \leqq \arg u_n < \pi$ and $1 \leqq i_1 < \dots < i_k \leqq n$ a sequence of integers. Then*

$$\arg u_{i_1} + \dots + \arg u_{i_k} = \sup_{\substack{M_1 \supset \dots \supset M_k \\ \dim M_p = n - i_p + 1}} \inf_{\xi_p \in M_p} (\arg a_1 + \dots + \arg a_k)$$

where $\{\xi_1, \dots, \xi_k\}$ is linearly independent and $a_1, \dots, a_k$ are proper values of the matrix $\big((A\xi_i, \xi_j)\big)$. Moreover the value $(\arg a_1 + \dots + \arg a_k)$ depends only on $[\xi_1, \dots, \xi_k]$.

Proof. The proof of this theorem is similar to the proof of theorem 6.

8. Definition. If $j_p \leqq i_p$ for $p = 1, \dots, k$, we write $(j_1, \dots, j_k) \leqq (i'_1, \dots, i'_k)$. Given any sequence $i_1 \leqq \dots \leqq i_k$ of integers such that $i_p \geqq p$, $p = 1, \dots, k$, let $(i'_1, \dots, i'_k)$ denote the *strictly increasing* sequence of positive integers such that

(a) $(i'_1, \dots, i'_k) \leqq (i_1, \dots, i_k)$,

(b) $(j_1, \dots, j_k) \leqq (i'_1, \dots, i'_k)$

wherever $(j_1, \dots, j_k)$ is a strictly increasing sequence of positive integers which is less than or equal to $(i_1, \dots, i_k)$. We observe that $(i'_1, \dots, i'_k)$ is given by the formula

$$i'_k = i_k$$

$$i'_p = \min(i_p, i_{p+1} - 1), \quad p = k-1, \dots, 1.$$

9. Theorem. *Let U be a unitary transformation on E_n with proper values $u_1, \dots, u_n$ such that $0 < \arg u_1 \leqq \dots \leqq \arg u_n < \pi$. Let $i_1 \leqq \dots \leqq i_k$ be a sequence of positive integers less than or equal to n such that $i_p \geqq p$.*

Then

$$\arg u_{i'_1} + \dots + \arg u_{i'_k} = \inf_{\substack{M_1 \subset \dots \subset M_k \\ \dim M_p = i_p}} \sup_{\substack{\xi_p \in M_p \\ \{\xi_p\} \text{ o.n.}}} \arg \det\big((U\xi_i, \xi_j)\big)$$

where $(i'_1, \dots, i'_k)$ is the sequence defined in § 8.

Proof. For subspaces $M_1 \subset \cdots \subset M_k$ with $\dim M_p = i_p$, $p = 1, \ldots, k$ there exists subspaces $M_1' \subset \cdots \subset M_k'$ with $M_p' \subset M_p$ and $\dim M_p' = i_p'$. Thus by § 5

$$\sup_{\substack{\xi_p \in M_p \\ \{\xi_p\} \text{ o.n.}}} \arg \det ((U\xi_i, \xi_j)) \geqq \sup_{\substack{\xi_p \in M_p' \\ \{\xi_p\} \text{ o.n.}}} \arg \det ((U\xi_i, \xi_j))$$

$$\geqq \arg u_{i_1}' + \cdots + \arg u_{i_k}'.$$

Now let $N_r = [\alpha_1, \ldots, \alpha_r]$, $r = 1, \ldots, n$, where $\{\alpha_i\}$ is an orthonormal set of proper vectors of U corresponding to $\{u_i\}$. Choose an orthonormal set $\{\delta_1, \ldots, \delta_k\}$ with $\delta_p \in N_{i_p}$, $p = 1, \ldots, k$ such that

$$\arg \det ((U\delta_i, \delta_j)) = \sup_{\substack{\xi_p \in N_{i_p} \\ \{\xi_p\} \text{ o.n.}}} \arg \det ((U\xi_i, \xi_j)).$$

By lemma 2.8 [1] there exists an orthonormal set $\{\eta_1, \ldots, \eta_k\}$ such that $\eta_p \in N_{i_p}$ and

$$\arg \det ((U\delta_i, \delta_j)) = \arg \det ((U\eta_i, \eta_j)).$$

But in § 5 we have proved that

$$\sup_{\substack{\xi_p \in N_{i_p'} \\ \{\xi_p\} \text{ o.n.}}} \arg \det ((U\xi_i, \xi_j)) = \arg u_{i_1}' + \cdots + \arg u_{i_k}'.$$

Therefore

$$\inf_{\substack{M_1 \subset \cdots \subset M_k \\ \dim M_p = i_p}} \sup_{\substack{\xi_p \in M_p \\ \{\xi_p\} \text{ o.n.}}} \arg \det ((U\xi_i, \xi_j)) \leqq \arg \det ((U\delta_i, \delta_j))$$

$$= \arg \det ((U\eta_i, \eta_j)) \leqq \arg u_{i_1}' + \cdots + \arg u_{i_k}'.$$

Thus the proof is complete.

10. Corollary. Let U satisfy the hypothesis of § 9 and let A be congruent to U. Let $i_1 \leqq \cdots \leqq i_k$ be a sequence of positive integers less than or equal to n such that $i_p \geqq p$. Then

$$\arg u_{i_1}' + \cdots + \arg u_{i_k}' = \inf_{\substack{M_1 \subset \cdots \subset M_k \\ \dim M_p = i_p}} \sup_{\xi_p \in M_p} (\arg a_1 + \cdots + \arg a_k),$$

where $\{\xi_1, \ldots, \xi_k\}$ ranges over linearly independent sets of vectors, a_i, $i = 1, \ldots, k$ are proper values of the matrix $((A\xi_i, \xi_j))$, and $(i_1', \ldots, i_k')$ is the sequence described in § 8. Moreover the values of $(\arg a_1 + \cdots + \arg a_k)$ depends only on $[\xi_1, \ldots, \xi_k]$.

Proof. We obtain 10 from 9 the same way as 6 was obtained from 5.

11. Theorem. *Let U be a unitary transformation on E_n and A be a linear transformation congruent to U. Let $u_1, \ldots, u_n$ be proper values of U such that* $0 < \arg u_1 \leqq \cdots \leqq \arg u_n < \pi$. Let $i_1, \ldots, i_k$ be a non-decreasing sequence of positive integers such that $i_p \leqq n - k + p$, $p = 1, \ldots, k$. Then

$$\arg u_{i_1}'' + \cdots + \arg u_{i_k}'' = \sup_{\substack{M_i \supset \cdots \supset M_k \\ \dim M_p = n - i_p + 1}} \inf_{\xi_p \in M_p} (\arg a_1 + \cdots + \arg a_k),$$

where $\{\xi_1, \dots, \xi_k\}$ *is linearly independent,* $a_1, \dots, a_k$ *are proper values of the matrix* $((A\xi_i, \xi_j))$*, and* $(i_1'', \dots, i_k'')$ *is the smallest strictly increasing sequence of integers which is* $\geqq (i_1, \dots, i_k)$ *in the sense of* § 8.

12. Remarks. The above propositions will be true if the strict inequality $<\pi$ changes to $\leqq \pi$.

Consider a unitary transformation U for which the condition $0 < \arg u_1 \leqq \dots \leqq \arg u_n \leqq \pi$ is not satisfied. Here $u_1, \dots, u_n$ are proper values of U. Suppose U does not have a proper value equal to 1. Then one can consider the unitary transformation V with proper values of $v_1, \dots, v_n$ where some of the v's are the same as u's and other v's are the same as $\bar{u}$'s such that $0 < \arg v_1 \leqq \dots \leqq \arg v_k \leqq \pi$. Then all the propositions may ba stated and proved for V. Thus a set of theorems can be obtained for U. The case the $u_1 = 1$ can be studied separately. We shall omit it.

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