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303. SOME FORMULAS OF HERMITE*

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1. The formulas

$$\sum_{m,\,n=0}^{\infty} \frac{m!\,n!}{(m+n+1)!}\,x^m y^n = -\frac{\log(1-x)(1-y)}{x+y-xy}, \tag{1.1}$$

$$\sum_{m,\,n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_m \left(\frac{1}{2}\right)_n}{(m+n)!}\,x^m y^n = \frac{(1-x)^{-\frac{1}{2}} + (1-y)^{-\frac{1}{2}}}{1+(1-x)^{\frac{1}{2}}(1-y)^{\frac{1}{2}}}, \tag{1.2}$$

where

$$(a)_n = a(a+1)\cdots(a+n-1),$$

are attributed to HERMITE by MARKOFF [**1**, p. 163].

MARKOFF proves (1.1) and (1.2) by means of finite differences. Slightly more general results of this kind can be proved rapidly by making use of the EULERian integral of the first kind. Let $\alpha>0$, $\beta>0$. Then it follows from

$$\frac{\Gamma(\alpha+m)\,\Gamma(\beta+n)}{\Gamma(\alpha+\beta+m+n)} = \int_0^1 t^{\alpha+m-1}(1-t)^{\beta+n-1}\,dt$$

that

$$\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \sum_{m,\,n=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(\alpha+\beta)_{m+n}}\,x^m y^n = \int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta-1}\,dt}{(1-tx)(1-(1-t)y)}$$

$$= \frac{1}{x+y-xy} \int_0^1 \left(\frac{x}{1-xt} + \frac{y}{1-y(1-t)}\right) t^{\alpha-1}(1-t)^{\beta-1}\,dt$$

$$= \frac{1}{x+y-xy} \left\{ \sum_{n=0}^{\infty} x^{n+1} \int_0^1 t^{\alpha+n-1}(1-t)^{\beta-1}\,dt + \sum_{n=0}^{\infty} y^{n+1} \int_0^1 t^{\alpha-1}(1-t)^{\beta+n-1}\,dt \right\}$$

$$= \frac{1}{x+y-xy} \left\{ \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\,\Gamma(\beta)}{\Gamma(\alpha+\beta+n)}\,x^{n+1} + \sum_{n=0}^{\infty} \frac{\Gamma(\alpha)\,\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}\,y^{n+1} \right\}.$$

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Therefore

$$(1.3)\qquad \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\beta)_n}{(\alpha+\beta)_{m+n}} x^m y^n = \frac{1}{x+y-xy}\left\{\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\alpha+\beta)_n}x^{n+1} + \sum_{n=0}^{\infty}\frac{(\beta)_n}{(\alpha+\beta)_n}y^{n+1}\right\}.$$

In particular, for $\alpha=\beta=1$, (1.3) becomes

$$\sum_{m,n=0}^{\infty}\frac{m!\,n!}{(m+n+1)!}x^m y^n = \frac{1}{x+y-xy}\left\{\sum_{n=1}^{\infty}\frac{x^n}{n}+\sum_{n=1}^{\infty}\frac{y^n}{n}\right\} = \frac{-\log(1-x)-\log(1-y)}{x+y-xy}.$$

For $\alpha=\beta=\frac{1}{2}$, (1,3) reduces to

$$\sum_{m,n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_m\left(\frac{1}{2}\right)_n}{(m+n)!}x^m y^n = \frac{1}{x+y-xy}\left\{\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{n!}x^{n+1}+\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{n!}y^{n+1}\right\}$$

$$= \frac{x(1-x)^{-\frac{1}{2}}+y(1-y)^{-\frac{1}{2}}}{x+y-xy}$$

$$= \frac{(1-x)^{-\frac{1}{2}}+(1-y)^{-\frac{1}{2}}}{1+(1-x)^{\frac{1}{2}}(1-y)^{\frac{1}{2}}}.$$

If we take $\beta=1-\alpha$, (1.3) becomes

$$\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(1-\alpha)_n}{(m+n)!}x^m y^n = \frac{1}{x+y-xy}\left\{\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}x^{n+1}+\sum_{n=0}^{\infty}\frac{(1-\alpha)_n}{n!}y^{n+1}\right\}$$

and therefore

$$(1.4)\qquad \sum_{m,n=0}^{\infty}\frac{(\alpha)_m(1-\alpha)_n}{(m+n)!}x^m y^n = \frac{1}{x+y-xy}\{x(1-x)^{-\alpha}+y(1-y)^{-1+\alpha}\}.$$

This reduces to (1.2) when $\alpha=\frac{1}{2}$.

2. The formula (1.3) can also be proved in the following way. Put

$$u(m,n)=\frac{(\alpha)_m(\beta)_n}{(\alpha+\beta)_{m+n}}.$$

Then, for $m>0$, $n>0$, we have

$$u(m-1,n)+u(m,n-1)-u(m-1,n-1)$$

$$=\frac{(\alpha)_{m-1}(\beta)_n}{(\alpha+\beta)_{m+n-1}}+\frac{(\alpha)_m(\beta)_{n-1}}{(\alpha+\beta)_{m+n-1}}-\frac{(\alpha)_{m-1}(\beta)_{n-1}}{(\alpha+\beta)_{m+n-2}}$$

$$=\frac{(\alpha)_{m-1}(\beta)_{n-1}}{(\alpha+\beta)_{m+n-1}}[(\alpha+m-1)+(\beta+n-1)-(\alpha+\beta+m+n-2)],$$

so that

(2.1) $$u(m-1,\, n)+u(m,\, n-1)-u(m-1,\, n-1)=0 \quad (m>0,\, n>0).$$

It follows that

$$\begin{aligned}(x+y-xy) \sum_{m,n=0}^{\infty} u(m,\, n)\, x^m y^n &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} u(m-1,\, n)\, x^m y^n + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} u(m,\, n-1)\, x^m y^n \\ &\qquad - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u(m-1,\, n-1)\, x^m y^n \\ &= \sum_{m=1}^{\infty} u(m-1,\, 0)\, x^m + \sum_{n=1}^{\infty} u(0,\, n-1)\, y^n \\ &= \sum_{m=1}^{\infty} \frac{(\alpha)_{m-1}}{(\alpha+\beta)_m} x^m + \sum_{n=1}^{\infty} \frac{(\beta)_{n-1}}{(\alpha+\beta)_n} y^n.\end{aligned}$$

This evidently proves (1.3).

3. Some special cases may be noted. If we take $y=x$, (1.3) becomes

(3.1) $$\sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\beta)_n}{(\alpha+\beta)_{m+n}} x^{m+n} = \frac{1}{2-x} \sum_{n=0}^{\infty} \frac{(\alpha)_n + (\beta)_n}{(\alpha+\beta)_n} x^n.$$

The left member of (3.1) is equal to

$$\sum_{n=0}^{\infty} \frac{x^n}{(\alpha+\beta)_n} \sum_{k=0}^{n} (\alpha)_k (\beta)_{n-k},$$

while the right member is equal to

$$\sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}} \sum_{k=0}^{+\infty} \frac{(\alpha)_n + (\beta)_n}{(\alpha+\beta)_n} x^n.$$

It follows that

(3.2) $$\frac{1}{(\alpha+\beta)_n} \sum_{k=0}^{n} (\alpha)_k (\beta)_{n-k} = \sum_{k=0}^{n} 2^{-n+k-1} \frac{(\alpha)_k + (\beta)_k}{(\alpha+\beta)_k}.$$

In particular, for $\alpha=\beta$, this reduces to

(3.3) $$\frac{1}{(2\alpha)_n} \sum_{k=0}^{n} (\alpha)_k (\alpha)_{n-k} = \sum_{k=0}^{\infty} 2^{-n+k} \frac{(\alpha)_k}{(2\alpha)_k}.$$

Thus, for $\alpha = 1,\ \frac{1}{2}$ we get

(3.4) $$\frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k}^{-1} = \sum_{k=0}^{n} \frac{2^{-n+k}}{k+1},$$

(3.5) $$\frac{1}{n!} \sum_{k=0}^{n} \left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{n-k} = 2^{-n} \sum_{k=0}^{n} \frac{1\cdot 3\cdot 5 \cdots (2k-1)}{k!}.$$

If we take $y=-x$ in (1.3), we get

$$(3.6)\qquad \sum_{m,n=0}^{\infty}(-1)^n\frac{(\alpha)_m(\beta)_n}{(\alpha+\beta)_{m+n}}x^{m+n}=\sum_{n=1}^{\infty}\frac{(\alpha)_n+(-1)^{n-1}(\beta)_n}{(\alpha+\beta)_n}x^{n-1}$$

$$=\sum_{n=0}^{\infty}\frac{(\alpha)_{n+1}+(-1)^n(\beta)_{n+1}}{(\alpha-\beta)_{n+1}}x^n.$$

This yields

$$(3.7)\qquad \sum_{k=0}^{n}(-1)^{n-k}(\alpha)_k(\beta)_{n-k}=\frac{(\alpha)_{n+1}+(-1)^n(\beta)_{n+1}}{\alpha+\beta+n}.$$

In particular, for $\alpha=\beta$, (3.7) gives

$$(3.8)\qquad \sum_{k=0}^{2n}(-1)^k(\alpha)_k(\alpha)_{2n-k}=\frac{(\alpha)_{2n+1}}{\alpha+n}.$$

For $\alpha=1,\ \frac{1}{2}$ we get

$$(3.9)\qquad \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^{-1}=\frac{2n+1}{n+1},$$

$$(3.10)\qquad \sum_{k=0}^{2n}(-1)^k\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_{2n-k}=\frac{2\left(\frac{1}{2}\right)_{2n+1}}{2n+1}.$$

4. To extend (1.3) to multiple sums we take

$$u(m,n,p)=\frac{(\alpha)_m(\beta)_n(\gamma)_p}{(\alpha+\beta+\gamma)_{m+n+p}}.$$

Then, for $m>0$, $n>0$, $p>0$,

$$u(m,n-1,p-1)+u(m-1,n,p-1)+u(m-1,n-1,p)-u(m-1,n-1,p-1)$$

$$=\frac{(\alpha)_{m-1}(\beta)_{n-1}(\gamma)_{p-1}}{(\alpha+\beta+\gamma)_{m+n+p-2}}[(\alpha+m-1)+(\beta+n-1)+(\gamma+p-1)$$

$$-(\alpha+\beta+\gamma+m+n+p-3)]=0.$$

Now put

$$(4.1)\qquad F_{\alpha,\beta,\gamma}(x,y,z)=\sum_{m,n,p=0}^{+\infty}u(m,n,p)x^my^nz^p.$$

Then

$$(yz+zx+xy-xyz)F_{\alpha,\beta,\gamma}(x,y,z)$$

$$=\sum_{m=0}^{\infty}\sum_{n,p=1}^{\infty}u(m,n-1,p-1)x^my^nz^p+\sum_{n=0}^{\infty}\sum_{m,p=1}^{\infty}u(m-1,n,p-1)x^my^nz^p$$

$$+\sum_{p=0}^{\infty}\sum_{m,n=1}^{\infty}u(m-1,n-1,p)x^my^nz^p-\sum_{m,n,p=1}^{\infty}u(m-1,n-1,p-1)x^my^nz^p$$

$$=\sum_{n,p=1}^{\infty}u(0,n-1,p-1)y^nz^p+\sum_{m,p=1}^{\infty}u(m-1,0,p-1)x^mz^p$$

$$+\sum_{m,n=1}^{\infty} u(m-1, n-1, 0)\, x^m y^n - \sum_{m,n,p=1}^{\infty} [u(m, n-1, p-1) + u(m-1, n, p-1)$$
$$+ u(m-1, n-1, p) - u(m-1, n-1, p-1)]\, x^m y^n z^p.$$

Hence if we define

$$F_{\alpha, \beta; \gamma}(x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m (\beta)_n}{(\alpha+\beta+\gamma)_{m+n}} x^m y^n, \tag{4.2}$$

it follows that

$$(yz + zx + xy - xyz)\, F_{\alpha, \beta, \gamma}(x, y, z) \tag{4.3}$$
$$= yz\, F_{\beta, \gamma; \alpha}(y, z) + zx\, F_{\gamma, \alpha; \beta}(z, x) + xy\, F_{\alpha, \beta; \gamma}(x, y).$$

Unfortunately (1.3) does not apply to $F_{\alpha, \beta; \gamma}(x, y)$, so that it does not seem possible to reduce (4.3) further. We note however that, by the method of § 2, we get

$$(x + y - xy)\, F_{\alpha, \beta; \gamma}(x, y) \tag{4.4}$$
$$= \sum_{n=1}^{\infty} \frac{(\alpha)_n}{(\alpha+\beta+\gamma)_n} x^n + \sum_{n=1}^{\infty} \frac{(\beta)_n}{(\alpha+\beta+\gamma)_n} y^n + \frac{\gamma xy}{\alpha+\beta+\gamma} F_{\alpha, \beta; \gamma+1}(x, y).$$

Iteration of (4.4) leads to the following formula

$$(x + y - xy)\, F_{\alpha, \beta; \gamma}(x, y) = \sum_{k=0}^{\infty} \frac{(\gamma)_k x^k y^k}{(x+y-xy)^k} \sum_{n=1}^{\infty} \frac{(\alpha)_n x^n + (\beta)_n y^n}{(\alpha+\beta+\gamma)_n}.$$

This may be rewritten in the form

$$(x + y - xy)\, F_{\alpha, \beta; \gamma}(x, y) = -2 \sum_{k=0}^{\infty} \frac{(\gamma)_n}{(\alpha+\beta+\gamma)_n} \left(\frac{xy}{x+y-xy}\right)^n \tag{4.5}$$
$$+ F_{\alpha, \gamma; \beta}\left(x, \frac{xy}{x+y-xy}\right) + F_{\beta, \gamma; \alpha}\left(y, \frac{xy}{x+y-xy}\right).$$

It is evident how a result of this kind can be obtained for the k-fold sum

$$\sum_{n_1, \ldots, n_k=0}^{\infty} u(n_1, \ldots, n_k)\, x_1^{n_1} \cdots x_k^{n_k},$$

where

$$u(n_1, \ldots, n_k) = \frac{(\alpha_1)_{n_1} \cdots (\alpha_k)_{n_k}}{(\alpha_1 + \cdots + \alpha_k)_{n_1 + \cdots + n_k}}.$$

REFERENCE

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