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279. AN INEQUALITY OF REDHEFFER*

Peter S. Bullen

0. In a very interesting paper [2] REDHEFFER gives a sharper form of the arithmetic-geometric inequality, inequality (6) below. In this note an inequality of RADO type, related to REDHEFFER'S inequality is given.

If $(d)=(d_1, d_2, \ldots)$, $(q)=(q_1, q_2, \ldots)$ are two sequences of positive numbers. The following notations will be used:

$$Q_n=\sum_{k=1}^{n} q_k, \qquad A_n=A_n(d;\, q)=\frac{1}{Q_n}\sum_{k=1}^{n} d_k\, q_k,$$

$$G_n=G_n(d;q)=\left(\prod_{k=1}^{n} d_k^{\,q_k}\right)^{1/Q_n}, \qquad \Gamma_n=A_n(G;\, q)=\frac{1}{Q_n}\sum_{k=1}^{n} G_k\, q_k.$$

The following inequalities are classical:

(1) $$A_n\geqslant G_n$$

with equality if and only if $a_1=\cdots=a_n$;

(2) $$\left(1-\frac{a}{b}\right)^{b} e^{a}\leqslant 1 \qquad (0\leqslant a\leqslant b)$$

with equality if and only if $a=0$.

1. Theorem. *If* $0\leqslant tq_n\leqslant Q_n$, *then*

(3) $$Q_n(A_n\, e^{-t}+t\,\Gamma_n-G_n)\geqslant Q_{n-1}(A_{n-1}\, e^{-t}+t\,\Gamma_{n-1}-G_{n-1})$$

with equality if and only if $t=0$ *and* $d_{n-1}=G_{n-1}$.

Proof **1.** Put $x=d_n$ and let $f(x)$ denote the right hand side of (3); then

$$f(x)=(Q_{n-1}A_{n-1}+q_n x)e^{-t}+tQ_{n-1}\Gamma_{n-1}-Q_n\left(1-\frac{q_n}{Q_n}\,t\right)G_{n-1}^{\frac{Q_{n-1}}{Q_n}}\,x^{\frac{q_n}{Q_n}}.$$

Simple calculations show f to have a single minimum at

$$x=x_0=\left(\left(1-\frac{q_n}{Q_n}\,t\right)e^{t}\right)^{Q_n/Q_{n-1}} G_{n-1},$$

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and that

$$f(x_0)=Q_{n-1}\left(A_{n-1}e^{-t}+t\,\Gamma_{n-1}-\left(1-\frac{q_n}{Q_n}t\right)^{Q_n/Q_{n-1}} e^{q_n t/Q_{n-1}}\,G_{n-1}\right).$$

By (2), and the hypothesis on t,

$$f(x_0)\geqslant Q_{n-1}(A_{n-1}\,e^{-t}+t\,\Gamma_{n-1}-G_{n-1}).$$

This completes the first proof of (3); the cases of equality are easily obtained.

***Proof* 2.** Rewrite (3) as

$$(A_n\,e^{-t}+t\,\Gamma_n-G_n)-\frac{Q_{n-1}}{Q_n}\,(A_{n-1}\,e^{-t}+t\,\Gamma_{n-1}-G_{n-1})\geqslant 0.$$

Easy calculations show that this last inequality is equivalent to

(4) $$\frac{q_n}{Q_n}\,d_n\,e^{-t}+\frac{Q_{n-1}}{Q_n}\,G_{n-1}\geqslant\left(1-\frac{q_n}{Q_n}\right)G_{n-1}{}^{Q_{n-1}/Q_n}\,d_n{}^{q_n/Q_n}\,.$$

By (1) the left-hand side of (4) is not less than

$$e^{-q_n t/Q_n}G_{n-1}{}^{Q_{n-1}/Q_n}\,d_n{}^{q_n/Q_n},$$

which by (2), and the hypothesis on t, is not less than the right-hand side of (4). This completes the second proof of (3), and again the cases of equality follow easily.

2. If $t=0$ the above theorem is an extension of RADO's inequality considered elsewhere, [**1**]; in particular (3) implies (1).

If $q_1=\cdots=q_n=1$ repeated applications of (3) shows that if $0\leqslant t\leqslant 2$,

5) $$G_n\leqslant Ae^{-t}+t\,\Gamma_n,$$

the inequality being strict unless $t=0$, and $d_1=\cdots=d_n$.

In [**2**] REDHEFFER shows that (5) holds for all $t\geqslant 0$, and by choosing the best value of t, given by $e^t=A_n/\Gamma_n$, deduces that

(6) $$e\,A_n\geqslant\Gamma_n\,e^{a_n/\Gamma_n}.$$

By an inequality of CARLEMAN this best value of t is not less than 1, but since it may exceed 2, we cannot deduce (6) from (3).

BIBLIOGRAPHY

[1] P. S. BULLEN, *Some inequalities for symmetric means*, Pac. J. Math. **15** (1965), 47—54.

[2] R. REDHEFFER, *Recurrent inequalities*, Proc. Lond. Math. Soc. (3) **17** (1967), 683—699.