PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE: MATHÉMATIQUES ET PHYSIQUE

№ 274 — № 301 (1969)

274. FROM THE HISTORY OF NONANALYTIC FUNCTIONS*

Dragoslav S. Mitrinović and *Jovan D. Kečkić*

A bibliographical note which, among other things, ascertains some priorities.

1. System of partial differential equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}, \tag{1}$$

where u and v are functions of x and y, subjected to certain conditions, is the simplest elliptic system, and, at the same time, presents the defining relation for analytic functions of a complex variable, namely $w(z)=u(x, y)+iv(x, y)$, with $z=x+iy$.

E. BELTRAMI (see: [1], [2]) was the first who gave the idea that, considering more complicated systems, other classes of complex functions can be defined. E. PICARD [3] also draw attention to that possibility, and studied systems of the form

$$\frac{\partial v}{\partial x}=a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y},\quad \frac{\partial v}{\partial y}=c\frac{\partial u}{\partial x}+d\frac{\partial u}{\partial y},$$

where a, b, c, d are functions of x and y, such that $(a-d)^2+4bc<0$.

It has become almost customary to ascribe to E. PICARD the priority of this idea (see, for example, [4], [5], [6]). The attention to this historical error was drawn by P. CARAMAN in his reviews [8] and [9].

The above idea was developed especially by L. BERS and A. GELBART, [10] and [11], I. N. VEKUA, [12], and G. N. POLOŽIĬ, [7], who introduced the so-called Σ-monogenic, p-analytic and p, q-analytic functions and generalised analytic functions by the defining equalities:

$$\sigma_1(x)\frac{\partial u}{\partial x}=\tau_1(y)\frac{\partial v}{\partial y},\quad \sigma_2(x)\frac{\partial u}{\partial y}=-\tau_2(x)\frac{\partial v}{\partial x}; \tag{Σ}$$

$$\frac{\partial u}{\partial x}=\frac{1}{p}\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{1}{p}\frac{\partial v}{\partial x}; \tag{p}$$

* Received September 15, 1969.

$$(p, q) \qquad p\frac{\partial u}{\partial x}+q\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}=0, \qquad -q\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0;$$

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=au+bv+f, \qquad \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=cu+dv+g,$$

where p, q, a, b, c, d, f, g in the above equalities are functions of x and y, subjected to certain conditions.

All those functions have a number of properties which are analogous to those of analytic functions. Properties of such functions are studied in detail in monographs [5], [7] and [12]. Dissertation [13] of D. S. DIMITROVSKI is also devoted to that subject.

2. D. POMPEIU, [14] and [15], looking for a formula which would be analogous to CAUCHY'S for analytic functions

$$f(\zeta)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(z)}{z-\zeta}\,dz,$$

arrived at the expression

$$\lim_{n\to\infty}\frac{1}{m(G_n)}\,\frac{1}{2i}\int_{\Gamma_n} w(z)\,dz,$$

where G_n and Γ_n are respectively sequences of regions, contours, inscribed respectively in the region G. contour Γ, which he called *areolare* (*areal*) *derivative* of a nonanalytic function. For analytic functions this expression clearly vanishes. The Romanian school, directed by D. POMPEIU, especially G. CALUGARÉANO ([16], [17], [18], [19]), M. NICOLESCO ([20]) and N. THÉODORESCU ([21], [22], [23], [24], [25]), gave a large number of important properties of this derivative, as well as of the integral

$$T_G f=-\frac{1}{\pi}\iint_G \frac{f(\zeta)}{\zeta-z}\,d\xi\,d\eta$$

which is inverse to it.

Remark. References [16] — [20] are taken from the paper [30] by E. R. HEDRICK, which will be mentioned later, while [21] — [25] are taken from [13].

Another generalisation was given by S. L. SOBOLEV [26] who introduced the following definition of a *generalised derivative*:

Let $f, g \in L(G)$, and let f and g satisfy the condition

$$\iint_G g\,\frac{\partial h}{\partial \bar z}\,dx\,dy+\iint_G fh\,dx\,dy=0,$$

where h is an arbitrary function, such that $h\in C(G)$, and that there exists a subset G_1 of G on which $h=0$. Then, we say that f is the generalised derivative of g with respect to $\bar z$.

I. N. VEKUA [27] found that SOBOLEV'S derivative coincides with POMPEIU'S if the first is continuous.

3. Starting with the expression

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{u_x + iv_x + m(u_y + iv_y)}{1 + im},$$

where $m = \frac{dy}{dx}$ is the slope of the path along which Δz approaches zero, a number of mathematicians in America, notably E. R. HEDRICK and E. KASNER, studied properties of nonanalytic functions, which they also called "polygenic" functions. Their research was mainly devoted to the geometrical interpretation of nonanalytic functions. So, for example, E. KASNER [28] has shown that the values $\frac{dw}{dz}$ for a fixed z, depending on m, lie on the circle

$$\left(\alpha - \frac{u_x + v_y}{2}\right)^2 + \left(\beta - \frac{v_x - u_y}{2}\right)^2 = \left(\frac{u_x - v_y}{2}\right)^2 + \left(\frac{v_x + u_y}{2}\right)^2,$$

where $\alpha + i\beta = \frac{dw}{dz}$. This circle clearly reduces to a point for analytic functions.

Paper [29] of E. R. HEDRICK is also partly devoted to the above KASNER *circle*.

They also defined and used operators $\mathcal{D}$, $\mathcal{P}$ given by

$$\mathcal{D}[f(z)] = \frac{1}{2}[u_x + v_y + i(v_x - u_y)],$$

$$\mathcal{P}[f(z)] = \frac{1}{2}[u_x - v_y + i(v_x + u_y)].$$

For other results in that direction consult HEDRICK'S expository article [30].

4. This theory of nonanalytic functions was preceded and developed in a different direction by G. V. KOLOSOV, [31], [32], [33]. He defines the operator D by

$$Dw = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),$$

(actually, in paper [32], this operator is denoted by D_{xy}), and uses it to integrate various systems of partial differential equations which arise in Mathematical Physics, especially in the Theory of Elasticity. KOLOSOV first proves in [31] certain formulas (KOLOSOV'S *formulas*) which enable him to work with D as if it were a derivative, and then he applies them in [33] in the way which is illustrated by the following example:

Linear equation of the type

$$a_0 D_n w + a_1 D_{n-1} w + \cdots + a_n w = 0,$$

where $a_0, \ldots, a_n$ are arbitrary analytic functions, and $D_2 w = D(Dw)$, etc., can be solved by analogy with the ordinary linear differential equation with constant coefficients

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0.$$

For example, if t_1 and t_2 are two different solutions of the equation

$$a_0 t^2 + a_1 t + a_2 = 0,$$

then the equation

$$a_0 D_2 w + a_1 Dw + a_2 w = 0, \tag{2}$$

has the following solution:

$$w = \alpha_1(z)\, e^{t_1 \bar{z}/2} + \alpha_2(z)\, e^{t_2 \bar{z}/2}, \tag{3}$$

where α_1, α_2 are arbitrary analytic functions.

Equations which contain operators $D, D_2, \ldots,$ KOLOSOV calls *conjugate equations*. Though, of course, KOLOSOV did not solve *every* conjugate equation, he clearly indicated a procedure which may be applied to them. Therefore, finding the solution of some special, particular, such equations would present nothing new.

However, KOLOSOV's results, though he published them several times in various journals, in Russia and abroad, were not sufficiently known to mathematicians. The reason is, perhaps, that his main results were given in articles which are, for their main part, devoted to the Theory of Elasticity, and such titles succeeded in hiding the mathematical theory which they included. (Paper **[33]** was, however, clearly reviewed in *Jahrbuch über die Fortschritte der Mathematik* **48** (1921—1922), 1402—1403, in the Section *Potentialtheorie und Theorie der partiellen Differentialgleichungen vom eliptischen Typus.*)

P. BURGATTI **[34]**, quotes KOLOSOV's paper **[32]** but again he considers the equation $D_n w = 0$.

Much later, A. BILIMOVIĆ (**[35]** — **[47]**) introduced the concept of *deviation from analyticity* as the vector

$$\vec{B} = \operatorname{grad} u + [\vec{k}, \operatorname{grad} v],$$

where $\vec{k} = [\vec{e_1}, \vec{e_2}]$ is the vector product of the unit vectors $\vec{e_1}, \vec{e_2}$ of orthogonal directions, and as the corresponding scalar

$$B = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).$$

He also introduced the operator $b = \dfrac{1}{2} B$.

BILIMOVIĆ mainly concentrates on the geometrical interpretation of those quantities, but he also shows how other definitions of POMPEIU, VEKUA, etc. can be expressed in terms of b (see particularly his expository articles **[45]**, **[46]** and **[47]**).

Clearly, BILIMOVIĆ'S operator B coincides with KOLOSOV'S operator D. Moreover, since BILIMOVIĆ concentrates mainly on the geometrical aspects of the theory of nonanalytic functions, it would be of interest to compare his results with those of HEDRICK and KASNER, who worked on similar lines.

Starting with the cited articles of A. BILIMOVIĆ, S. FEMPL, **[48]** — **[55]**, solves special types of equations which involve the operator B. However, G. V. KOLOSOV has either directly solved those equations, or has indicated how they may be solved. Papers **[48]** — **[55]** of S. FEMPL present, therefore, nothing new in idea, as they are only special cases of more general results obtained by KOLOSOV. The same applies to the article **[56]** of J. D. KEČKIĆ, who gave a generalisation of a FEMPL'S theorem proved in **[54]**.

Remark. In solving equation (2) KOLOSOV made a slight error, stating that its solution is given by $w = a_1(z)\, e^{t_1 \bar{z}} + a_2(z)\, e^{t_2 \bar{z}}$. The correct solution (3) was obtained in articles **[54]** and **[56]**.

It should be noted here that none of the reviews of papers **[48]** — **[53]** mentions the previous results of KOLOSOV. In review **[57]**, T. LESER mentions what is essentially the same result obtained by N. THÉODORESCU **[58]**, but does not appear to see the connection between it and FEMPL'S results. The only exception presents review **[59]** of paper **[54]**, but again, the reviewer, though he calls FEMPL'S results "wohlbekannten", does not indicate where they have been published before.

5. B. RIEMANN arrived at system (1) in the following way: In his dissertation **[60]** he gives the formula

$$\text{(4)}\qquad \frac{dw}{dz} = \frac{1}{2}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] + \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] \cdot e^{-2i\varphi},$$

where $dz = \varepsilon e^{i\varphi}$, and (1) presents the condition that $\dfrac{dw}{dz}$ does not depend on the direction φ.

Introducing the notations

$$\text{(5)}\qquad \frac{\partial w}{\partial z} = \frac{1}{2}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right], \quad \frac{\partial w}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right],$$

expression (4) can be written in the form

$$dw = \frac{\partial w}{\partial z}\, dz + \frac{\partial w}{\partial \bar{z}}\, d\bar{z},$$

as quoted by L. HÖRMANDER at the beginning of his book **[61]**.

The complex form of system (1) is, therefore, $\dfrac{\partial w}{\partial \bar{z}} = 0$. For other consequences of formula (4) see **[62]**.

Notations (5) seem to be first used by W. WIRTINGER **[63]** in 1927. We have not found them in literature before him, and, besides, S. BERGMAN in his book **[64]** also says that they were introduced by WIRTINGER, and applies them to some problems in Fluid Dynamics.

For the application of those operators to partial differential equations, see **[65]**.

I. N. VEKUA, in the cited monograph **[12]**, also uses operators $\frac{\partial w}{\partial z}, \frac{\partial w}{\partial \bar{z}}$ and says that they can be treated formally as derivatives. He also uses the notation $\partial_z w$ and $\partial_{\bar{z}} w$.

6. From the above exposition it follows that all the cited operators introduced for nonanalytic functions are, in fact, either KOLOSOV's operator D, or a constant multiple of it. We can, therefore, say that G. V. KOLOSOV was the first to use those operators for integration of elliptic systems of partial differential equations. It is surprising that his results were not sufficiently known, and that even some Russian mathematicians make no mention of him. A typical example of this ignorance of KOLOSOV's results is provided by monograph **[66]**, where the apparatus of complex functions is also used, but on a much lower level than had previously been done by KOLOSOV. In fact, in all the above articles or books we have only found his paper **[31]** referred to in **[7]** and **[13]**, and paper **[32]** in **[34]**. Nevertheless, the priority and the credit for the above method of integration undoubtedly go to him.

We have given, in this article, only an outline of the development of the theory of nonanalytic functions, as our aim was to ascertain the priority of those discoveries. We hope that this article will serve to prevent further rediscoveries in that field, as well as to initiate other developments of the theory of nonanalytic functions. Many details which we have omitted here, can be found in **[7]**, **[12]**, **[13]**, **[30]**, **[45]**, **[46]** and **[47]**.

REFERENCES

[1] E. BELTRAMI, *Delle variabili complesse sopra una superficie qualunque*, Ann. Mat. pura appl. (1) **2** (1867/68), 329—366.

[2] E. BELTRAMI, *Sulle funzioni potenziali di sistemi intorno ad un asse*, Rend. Ist. Lombardo Sci. Lett. (11) **2** (1878), 668—680.

[3] E. PICARD, *Sur une généralisation des équations de la théorie des fonctions d'une variable complexe*, C. R. Acad. Sci. Paris **112** (1891), 1399—1403.

[4] L. BERS, *An outline of the theory of pseudoanalytic functions*, Bull. Amer. Math. Soc. **62** (1956), 291—331.

[5] L. BERS, *Theory of pseudoanalytic functions*, New York 1953.

[6] I. CRISTEA, *An extension of a conformal transformation in three dimensional space* (Romanian), Studii Cerc. Mat. **16** (1964), 1033—1057.

[7] Г. Н. Положий, *Обобщение теории аналитических функций комплексного переменного*, Издательство Киевскога Университета 1965.

[8] P. CARAMAN, *Review*, Zentralblatt für Mathematik **121** (1966), 65—66.

[9] P. CARAMAN, *Review*, ibid., **133** (1967), 330—331.

[10] L. BERS and A. GELBART, *On a class of functions defined by partial differential equations*, Trans. Amer. Math. Soc. **56** (1944), 67—93.

[11] L. BERS, *Partial differential equations and generalised analytic functions*, Proc. Nat. Acad. Sci. **36** (1950), 130—136 and **37** (1951), 42—47.

[12] И. Н. ВЕКУА, *Обобщенные аналитические функции*, Москва 1959.

[13] D. S. DIMITROVSKI, *Prilog kon teorijata na obopštenite analitički funkcii*, Dissertation, unpublished.

[14] D. POMPEIU, *Sur la continuité des fonctions d'une variable complexe*, Thesis, Paris 1905.

[15] D. POMPEIU, *Sur une classe des fonctions d'une variable complexe*, Rend. Circ. Mat. Palermo **33** (1912), 108—113 and **35** (1913), 277—281.

[16] G. CALUGARÉANO, *Sur les fonctions polygènes d'une variable complexe*, Thesis, Paris 1928.

[17] G. CALUGARÉANO, *Sur une classe d'équations du second ordre intégrables à l'aide des fonctions polygènes*, C. R. Acad. Sci. Paris **186** (1928), 1406—1407.

[18] G. CALUGARÉANO, *Les fonctions polygènes comme intégrales d'équations différentielles*, Trans. Amer. Math. Soc. **31** (1929), 372—378.

[19] G. CALUGARÉANO, *On differential equations admitting polygenic integrals*, ibid. **32** (1930), 110—113.

[20] M. NICOLESCO, *Fonctions complexes dans le plan et dans l'espace*, Thesis, Paris 1928.

[21] N. THÉODORESCU, *La dérivée aréolaire et ses applications physiques*, Thesis, Paris 1931.

[22] N. THÉODORESCU, *Cercetarile romanesti si sovietice in teoria derivatei areolare si a functiilor monogene a*, Annalele roumano-sovietice, Ser. Math. — Fiz. **2** (1958), 5—20.

[23] N. THÉODORESCU, *Dérivée et primitives aréolaires*, Annali di mathematica **44** (1960).

[24] N. THÉODORESCU, *Dérivée aréolaire globale et dérivée généralisée*, Bull. Math. de la Soc. de Sci. Math. de la R. P. R. **6** (1962), 3—4.

[25] N. THÉODORESCU, *Méthodes fonctionnelles en théorie des fonctions d'une variable complexe*, Bull. math. Soc. Sci. Math. Phys. **5** (1964), 225—264.

[26] С. Л. СОБОЛЕВ, *Некоторые применения функционального анализа в математической физике*, ЛГУ 1950.

[27] И. Н. ВЕКУА, *Общее представление функций двух независимых переменных допускающих производные в смысле С. Л. Соболева и проблема примитивных*, ДАН **89** (1953), 773—775б.

[28] E. KASNER, *A new theory of polygenic (or non-monogenic) functions*, Science, **46** (1927), 581—582.

[29] E. R. HEDRICK, *On the derivatives of non-analytic functions*, Proc. Nat. Acad. **14** (1928), 649—654.

[30] E. R. HEDRICK, *Non-analytic functions of a complex variable*, Bull. Amer. Math. Soc. **39** (1933), 75—96.

[31] Г. В. КОЛОСОВ, *Об одном приложении теории функций комплексного переменного к плоской задаче математической теории упругости*, Юрьев 1909.

[32] G. KOLOSSOFF, *Über einige Eigenschaften des ebenen Problems der Elastizitätstheorie*, Z. Math. Phys. **62** (1914), 384—409.

[33] Г. В. КОЛОСОВ, *О сопряженныхъ дифференцiальныхъ уравненiяхъ съ частными производными съ приложенiемъ ихъ къ рѣшенiю вопросовъ математической физики*, Ann. Inst. électrot. Petrograd **11** (1914), 179—199.

[34] P. BURGATTI, *Sulle funzioni analitiche d'ordine n*, Boll. Un. Mat. Ital. **1** (1922), 8—12.

[35] A. BILIMOVITCH, *Sur la mesure de déflexion d'une fonction non-analytique par rapport à une fonction analytique*, C. R. Acad. Sci. Paris **237** (1953), 694.

[36] A. BILIMOVIĆ, *O meri odstupanja neanalitičke funkcije od analitičnosti*, Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka **CCXXI** (1956), 1—11.

[37] A. BILIMOVIĆ, *Afina transformacija neanalitičke funkcije u analitičku*, ibid. **CCXXI** (1956), 13—17.

[38] A. BILIMOVIĆ, *O dijagramu neanalitičke funkcije za datu tačku*, ibid. **CCXXI** (1956), 39—43.

[39] A. BILIMOVITCH, *Application en hydromécanique de la mesure de déflexion d'analyticité d'une function non-analytique*, Bull. Acad. Serbe Sci. Arts Cl. Sci. Math. Natur. Sci. Math. **10** (1956), 33—41.

[40] A. BILIMOVIĆ, *Otstupanje neanalitičke kvaternion-funkcije od analitičnosti*, Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka **CCXXVIII** (1957), 1—22.

[41] A. Bilimovitch, *Sur la déflexion d'une fonction non-analytique du quaternion par rapport à une fonction analytique*, Bull. Acad. Serbe Sci. Arts Cl. Sci. Math. Natur. Sci. Math. **20** (1957), 1—9.

[42] A. Bilimovitch, *Sur les transformations des fonctions non analytiques*, C. R. Acad. Sci. Paris **247** (1958), 1954.

[43] A. Bilimovitch, *Sur la mesure de déflexion d'une fonction non-analytique par rapport à une fonction analytique*, Publ. Inst. Math. (Beograd) **6** (1954), 17—26.

[44] A. Bilimovitch, *Sur les lignes principales des fonctions non analytiques*, C. R. Acad. Sci. Paris **250** (1960), 805—807.

[45] A. Bilimović, *Diferencijalni elementi geometrijske teorije neanalitičkih funkcija*, Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka **CCXLII** (1960), 1—82.

[46] A. Bilimović, *O nekim integralnim teoremama u geometrijskoj teoriji neanalitičkih funkcija*, ibid. **CCLXIII** (1966), 53—81.

[47] A. Bilimović, *Delimične diferencijalne jednačine u geometrijskoj teoriji neanalitičkih funkcija*, ibid. **CCLXIX** (1967), 79—109.

[48] S. Fempl, *O neanalitičkim funkcijama čije je odstupanje od analitičnosti analitička funkcija*, ibid. **CCLIV** (1963), 75—80.

[49] S. Fempl, *O neanalitičkim funkcijama čije je drugo odstupanje od analitičnosti analitička funkcija*, Bull. Soc. Math. Phys. Serbie **XV** (1963), 57—62.

[50] S. Fempl, *Areolarni polinomi kao klasa neanalitičkih funkcija čiji su realni i imaginarni delovi poliharmonijske funkcije*, Mat. Vesnik **1** (16) (1964), 29—38.

[51] S. Fempl, *Reguläre Lösungen eines Systems partieller Differentialgleichungen*, Publ. Inst. Math. (Beograd) **4** (18) (1964), 115—120.

[52] С. Фемпл, *Об одной системе уравнений в частных производных, решение которой приводится к интегрованию урвнения вида Клеро*, Differencial'nye Uravneniya **1** (1965), 698—700.

[53] S. Fempl, *Über einige Systeme partieller Differentialgleichungen*, These Publications **№ 143— № 155** (1965), 9—12.

[54] S. Fempl, *Areoläre Exponentialfunktion als Lösung einer Klasse Differentialgleichungen*, Publ. Inst. Math. (Beograd) **8** (22) (1968), 138—142.

[55] S. Fempl, *Über eine partielle Differentialgleichung in der nicht analytische Funktionen erscheinen*, ibid. **9** (23) (1969), 115—122.

[56] J. D. Kečkić, *O jednoj klasi parcijalnih jednačina*, Mat. Vesnik **6** (21) (1969), 71—73

[57] T. Leser, *Review 6128*, Math. Reviews **34** (1967), 1119.

[58] N. Théodorescu, *La derivée aréolaire*, Ann. Roum. Math. **3** (1936), 3—62.

[59] K. Lohmann, *Review*, Zentralblatt für Mathematik **162** (1969), 110.

[60] Б. Риман, *Сочинения*, Москва 1948.

[61] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, Princeton 1966.

[62] J. D. Kečkić, *Analytic and c-analytic functions*, Publ. Inst. Math. (Beograd) **9** (23) (1969), 189—198.

[63] W. Wirtinger, *Zur formalen Theorie der Funktionen von mehr komplexen Veränderlichen*, Math. Ann. **97** (1927), 357—375.

[64] S. Bergman, *Kernel Functions and Elliptic Differential Equations in Mathematical Physics*, New York 1953.

[65] S. Bergman, *Integral Operators in the Theory of Linear Partial Differential Equations*, Berlin—Göttingen—Heidelberg 1961.

[66] E. G. Coker and L. N. G. Filon, *A treatise on Photo-elasticity*, Cambridge 1931.