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256. ON SOME SYSTEMS OF PARTIAL DIFFERENTIAL EQUATION AND ON SOME CLASSES OF NON-ANALYTIC FUNCTIONS*

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In the first part regular solutions [1] of the system

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=f(x, y, u, v); \quad \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=g(x, y, u, v),$$

for some special cases, are obtained. In the second part a system of partial equations of second order is solved. In the third part two classes of non-analytic functions are given in a closed form.

1.1. In this part we shall solve the following system of partial differential equations

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=f(x, y, u, v); \quad \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=g(x, y, u, v)$$

$(u=u(x, y),\ v=v(x, y))$ for some special cases.

Multiplying the second equation of the system by i and adding it to the first, the left hand side becomes

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}+i\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right).$$

This expression represents the *areolare derivative* of POMPEIU [2] multiplied by 2, or the operator B,

$$B=\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}$$

which was introduced by BILIMOVIĆ, who called it *deviation from being analytic* [3].

Besides the operator B, in this paper we shall use its inverse operator $\mathcal{S}$, introduced by FEMPL. In that case $\mathcal{S}\,\Phi(z, \bar{z})=w$ denotes that $Bw=\Phi(z, \bar{z})$.

The following propreties of the operators B and $\mathcal{S}$ can be easily checked:

$$B(w_1+w_2)=Bw_1+Bw_2, \quad Bw_1w_2=w_1Bw_2+w_2Bw_1,$$

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$Bw=0$ if and only if w is an analytic function,

$$Bf(w)=f'(w)\,Bw,\quad B\bar{z}=2,$$

$$\mathcal{S} f(z, w)\,Bw=\int f(z, w)\,dw+\alpha(z),\quad \mathcal{S} f(z, \bar{z})=\frac{1}{2}\int f(z, \bar{z})\,d\bar{z}+\alpha(z),$$

where in the last two relations $\alpha(z)$ represents an arbitrary analytic function.

1.2. Consider the following system

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=\operatorname{Re} f\left(z, \frac{w}{\bar{z}}\right),\quad \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=\operatorname{Im} f\left(z, \frac{w}{\bar{z}}\right), \tag{1}$$

where $w=u+iv$, $z=x+iy$, $\bar{z}=x-iy$.

Multiplying the second equation by i and adding it to the first, we get

$$Bw=f\left(z, \frac{w}{\bar{z}}\right) \tag{2}$$

$\left(\text{The equation } Bw=f\left(\frac{w}{\bar{z}}\right) \text{ has been solved by S. Fempl [4]}\right)$.

Introduce the following substitution: $w=t\bar{z}$. Then $Bw=2t+\bar{z}Bt$ and equation (2) becomes

$$2t+\bar{z}Bt=f(z, t),$$

i.e., $\dfrac{Bt}{f(z, t)-2t}=\dfrac{1}{\bar{z}}$, i.e., $\mathcal{S}\dfrac{Bt}{f(z, t)-2t}=\mathcal{S}\dfrac{1}{\bar{z}}$.

Using the properties of the operator $\mathcal{S}$, we get

$$\int\frac{1}{f(z, t)-2t}\,dt=\frac{1}{2}\int\frac{1}{\bar{z}}\,d\bar{z}+P(z),$$

where $P(z)$ is an arbitrary analytic function.

Let $\Phi(z, t)=\displaystyle\int\frac{1}{f(z, t)-2t}\,dt$. Then $\bar{z}=\alpha(z)\,e^{2\Phi(z, t)}$, where $\alpha(z)=e^{-2P(z)}$ is also an analytic function, is the solution of (2).

Therefore

$$x=\operatorname{Re}[\alpha(z)\,e^{2\Phi(z, t)}],\quad y=-\operatorname{Im}[\alpha(z)\,e^{2\Phi(z, t)}]$$

is the regular solution [1] of the system (1).

1.3. Let

$$\begin{aligned}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}&=\operatorname{Re} f\left(\frac{a(z)\,\bar{z}+b(z)\,w+c(z)}{A(z)\,\bar{z}+B(z)\,w+C(z)}\right),\\ \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}&=\operatorname{Im} f\left(\frac{a(z)\,\bar{z}+b(z)\,w+c(z)}{A(z)\,\bar{z}+B(z)\,w+C(z)}\right),\end{aligned} \tag{3}$$

where a, b, c, A, B, C are analytic functions, and let $\begin{vmatrix} a(z) & b(z) \\ A(z) & B(z)\end{vmatrix}\neq 0$.

System (3) is equivalent to

$$(4) \qquad Bw = f\left(\frac{a(z)\bar{z} + b(z)w + c(z)}{A(z)\bar{z} + B(z)w + C(z)}\right).$$

The solution of (4) is

$$(5) \qquad w = \beta(z) + \Phi(\bar{z} - \alpha(z), z),$$

where $t = \Phi(\zeta, z)$ is the solution of

$$(6) \qquad Bt = f\left(\frac{a(z)\zeta + b(z)t}{A(z)\zeta + B(z)t}\right) = F\left(z, \frac{t}{\zeta}\right),$$

$$(7) \qquad \zeta = \bar{z} - \alpha(z), \quad t = w - \beta(z),$$

and α, β are solutions of the system

$$(8) \qquad a(z)\alpha + b(z)\beta + c(z) = 0, \quad A(z)\alpha + B(z)\beta + C(z) = 0.$$

Proof. Let us show first that α, β are analytic functions. Apply the operator B on the system (8), and we get

$$a(z)B\alpha + b(z)B\beta = 0, \quad A(z)B\alpha + B(z)B\beta = 0.$$

Since the determinant of the above system is not zero, its only solutions are trivial $B\alpha = B\beta = 0$, i.e., α and β are analytic functions.

Let $t = \Phi(\zeta, z)$ be the solution of the equation (6). Then

$$Bt = 2\Phi_1'(\zeta, z) = f\left(\frac{a(z)\zeta + b(z)t}{A(z)\zeta + B(z)t}\right)$$

where Φ_1' denotes the partial derivative with respect to the first argument

However, starting from (5) we get $Bw = 2\Phi_1'(\bar{z} - \alpha(z), z)$, and according to (7): $Bt = Bw$.

Using (7), (8), we conclude that

$$\frac{a(z)\zeta + b(z)t}{A(z)\zeta + B(z)t} = \frac{a(z)\bar{z} + b(z)w + c(z)}{A(z)\bar{z} + B(z)w + C(z)}$$

and (5) is, therefore the solution of the equation (4).

In other words,

$$u = \mathrm{Re}\,[\beta(z) + \Phi(\bar{z} - \alpha(z), z)], \quad v = \mathrm{Im}\,[\beta(z) + \Phi(\bar{z} - \alpha(z), z)]$$

is the regular solution of the system (3).

1.4. The system

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \mathrm{Re}\,[f(z, w)\,g(z, \bar{z})], \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \mathrm{Im}\,[f(z, w)\,g(z, \bar{z})]$$

has the following complex form

$$(9) \qquad Bw = f(z, w)\,g(z, \bar{z}), \quad \text{i.e.,} \quad \frac{Bw}{f(z, w)} = g(z, \bar{z}).$$

(A special case of this equation, $Bw = f(w)\,g(\bar{z})$, has been solved in **[3]**;

Considering the properties of the operator B, we have

$$\int \frac{1}{f(z, w)}\, dw = \frac{1}{2} \int g(z, \bar{z})\, d\bar{z} + P(z),$$

where $P(z)$ is an analytic function.

The above expression represents the solution of the equation (9).

EXAMPLE. The system

(10) $$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{(x^2 - y^2) u + 2xyv}{x^2 + y^2}, \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{(x^2 - y^2) v - 2xyu}{x^2 + y^2}$$

becomes

$$Bw = \frac{w\bar{z}}{z}$$

and the solution is, therefore,

$$\int \frac{dw}{w} = \frac{1}{2} \int \frac{\bar{z}}{z}\, d\bar{z} + P(z).$$

After some rearranging, and putting $P_1(z) = \log P(z)$, we get

$$w = P_1(z) \exp\left(\frac{1}{4} \frac{\bar{z}^2}{z}\right).$$

If $P_1(z) = \alpha(x, y) + i\beta(x, y)$, we have

$$u(x, y) = \exp\left(\frac{x^3 - 3xy^2}{4(x^2 + y^2)}\right)\left[\alpha(x, y) \cos \frac{3x^2y - y^3}{4(x^2 + y^2)} + \beta(x, y) \sin \frac{3x^2y - y^3}{4(x^2 + y^2)}\right];$$

$$v(x, y) = \exp\left(\frac{x^3 - 3xy^2}{4(x^2 + y^2)}\right)\left[\beta(x, y) \cos \frac{3x^2y - y^3}{4(x^2 + y^2)} - \alpha(x, y) \sin \frac{3x^2y - y^3}{4(x^2 + y^2)}\right],$$

and this is the regular solution of the system (10).

2. Starting from the operator B, operators of higher order can be defined:

$$B_1 \overset{\text{def}}{=} B, \quad B_{n+1} \overset{\text{def}}{=} B(B_n), \qquad n > 1.$$

For the complex function w, the expression $B_n w$ is called *n-th deviation from being analytic* (*n-th deviation*).

In this part we shall solve an equation which involves the operator B_2. Let

(11) $$B_2 w = f(z, w).$$

Then

(12) $$\int \frac{dw}{\sqrt{2 \int f(z, w)\, dw + \alpha(z)}} = \frac{\bar{z}}{2} + \beta(z)$$

is the solution of the equation (11), where α, β are arbitrary analytic functions.

Proof. Using the properties of the operator $\mathcal{S}$, we see that (12) can be written in the form

(13) $$\mathcal{S} \frac{Bw}{\sqrt{2 \mathcal{S} f(z, w)\, Bw + \alpha(z)}} = \frac{\bar{z}}{2} + \beta(z).$$

Applying the operator B on the equation (13), we get

(14) $$\frac{Bw}{\sqrt{2\mathcal{S}f(z,w)\,Bw+a(z)}}=1,\quad \text{i.e.,}\quad Bw=\sqrt{2\mathcal{S}f(z,w)\,Bw+a(z)}.$$

It then follows that

$$B_2w=\frac{2f(z,w)\,Bw}{2\sqrt{2\mathcal{S}f(z,w)\,Bw+a(z)}}$$

or, using (14),

$$B_2w=f(z,w).$$

The given solution of the equation (11) facilitates the method of solving the system of second order partial differential equation of the form

$$\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 v}{\partial x\,\partial y}-\frac{\partial^2 u}{\partial y^2}=\operatorname{Re} f(z,w),\qquad \frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 u}{\partial x\,\partial y}-\frac{\partial^2 v}{\partial y^2}=\operatorname{Im} f(z,w).$$

EXAMPLE. System

(15) $$\begin{aligned}&\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}=(x^2-y^2)\,e^{-2u}\cos 2v+2xye^{-2u}\sin 2v\\ &\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 v}{\partial y^2}=2xye^{-2u}\cos 2v-(x^2-y^2)\,e^{-2u}\sin 2v\end{aligned}$$

becomes

(16) $$e^{2w}B_2w=z^2,\quad \text{where}\quad w=u+iv,\quad z=x+iy.$$

The solution of (16) is

$$\varphi_1(z)\,e^w=z\operatorname{ch}\left[\varphi_1(z)\frac{\bar z}{2}+\varphi_2(z)\right],\qquad (\varphi_1,\ \varphi_2 \text{ arbitrary analytic functions})$$

and, therefore,

$$u=\operatorname{Re}\log\frac{z}{\varphi_1(z)}\operatorname{ch}\left[\varphi_1(z)\frac{\bar z}{2}+\varphi_2(z)\right],\qquad v=\operatorname{Im}\log\frac{z}{\varphi_1(z)}\operatorname{ch}\left[\varphi_1(z)\frac{\bar z}{2}+\varphi_2(z)\right]$$

represents the solution of the system (15).

3. Finally, we give two classes of non-analytic functions in a closed form.

3.1. Any non-analytic function of the form

(17) $$w=\varphi(z)+\sum_{\nu=1}^{n}\varphi_\nu(z)\,e^{\frac{a_\nu}{2}\bar z},$$

where $\varphi_\nu(z)$ are arbitrary analytic functions, and a_ν ($\nu=1,\ldots,n$) are n-th roots of unity has the property that the difference between it and its n-th deviation is an analytic function $\varphi(z)$.

Proof. The following equation is to be solved:

(18) $$w-B_nw=\varphi(z)$$

Since the general solution of the equation $w-B_nw=0$ is [5]

$$w=\sum_{\nu=1}^{n}\varphi_\nu(z)\,e^{\frac{a_\nu}{2}\bar z}$$

using the method given in [6] we see that the general solution of (18) is (17).

3.2. Any non-analytic function of the form

$$(19) \qquad w=\sum_{\nu=1}^{n} P_\nu(z)\exp\left(\frac{\bar{z}}{2\,\alpha_\nu(z)}\right)$$

where $P_\nu(z)$ are arbitrary analytic functions, and $\alpha_\nu(z)$ $(\nu=1, \ldots, n)$ are n branches of the function $\sqrt[n]{\varphi(z)}$, has the property that the quotient between it and its n-th deviation is an analytic function $\varphi(z)$.

Proof. The following equation is to be solved

$$\frac{w}{B_n w}=\varphi(z),$$

i.e.,

$$(20) \qquad B_n w-\frac{1}{\varphi(z)}\,w=0.$$

This type of equation has been solved in [7], from where it can be seen that (19) is the general solution of the equation (20).

In the case when $n=1$, both given classes of non-analytic functions have been determined by S. Fempl [8].

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