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252. MONOTONY AND THE BEST POSSIBLE BOUNDS OF SOME SEQUENCES OF SUMS*

Dušan D. Adamović and *Milan R. Tasković*

0. Prof. D. S. Mitrinović has proposed in [1] the following problem:

"*Determine some sharp enough bounds of*

$$\frac{1}{pn+1}+\frac{1}{pn+2}+\cdots+\frac{1}{qn+1}\qquad (n=1,2,\ldots),$$

where p and q are fixed natural numbers, and if possible, the best ones."

By using Cauchy-Schwarz's and Schweizer's inequalities, A. Lupas, in his solution of this problem [2], gives for the expression

$$(1)\qquad S_n(p,q)\overset{\text{def}}{=}\sum_{k=pn+1}^{qn+1}\frac{1}{k}\qquad (p<q;\ p,\ q,\ n \text{ natural numbers})$$

the following estimate

$$(2)\qquad \underline{L}_n(p,q)\overset{\text{def}}{=}2\,\frac{(q-p)\,n+1}{(q+p)\,n+2}<S_n(p,q)<\frac{[(q-p)\,n+1]\,[(q+p)\,n+2]}{2\,(pn+1)\,(qn+1)}\overset{\text{def}}{=}\overline{L}_n(p,q)$$

$$(p<q;\ p,\ q,\ n \text{ natural numbers}),$$

which follows from his more general estimate

$$(3)\qquad \frac{t}{A(s,s+t)}<\sum_{k=s+1}^{s+t}\frac{1}{k}<\frac{t}{H(s,s+t)}\qquad (s,\ t \text{ natural numbers}),$$

where $A(a,b)$ and $H(a,b)$ denote arithmetic and harmonic means of numbers a and b respectively.

In Lupas' solution the question of the best possible bounds was not treated. In fact in the case when the bounds depend of p, q and n, as in Lupas' estimate, the problem of the best possible bounds does not have a real meaning, because the unique solution for the bounds (for both of them) is the expression $S_n(p,q)$.

On the other hand, if $S_n(p,q)$ is regarded as a sequence determined by parameters p and q, and the bounds are given as the functions of p and q only, the mentioned question has a non trivial meaning and is without any ambiguity: the lower bound $\underline{B}(p,q)$ and the upper bound $\overline{B}(p,q)$ are the best possible if and only if the following conditions hold:

$$\underline{B}(p,q)=\operatorname*{Inf}_{n\geqslant 1} S_n(p,q),\quad \overline{B}(p,q)=\operatorname*{Sup}_{n\geqslant 1} S_n(p,q).$$

* Presented January 25, 1969 by D. S. Mitrinović.

1. The best possible bounds of the sum (2), in the previous sense, for all natural numbers p and q (with $p<q$), save for a fixed set of pairs, are given in Proposition 2. Paralelly with sums (1), we also considered the following sums

$$\sigma_n(p, q) \overset{\text{def}}{=} \sum_{k=pn+1}^{qn} \frac{1}{k} \qquad (p<q;\ p,\ q,\ n \text{ natural numbers}). \tag{4}$$

The best possible bounds $\underline{\mathcal{B}}(p, q)$ and $\overline{\mathcal{B}}(p, q)$ of sums (4), and in the same sense, gives Proposition 1.

In Propositions 1 and 2 the monotony of sequences $\sigma_n(p, q)$ and $S_n(p, q)$ is examined too. This monotony is applied in the determination of the best possible bounds, and is interesting by itself.

We note that a solution of this problem of Prof. D. S. Mitrinović, given by us in [2], contains an error that we have made carrying over the result obtained for $S_n(p, p+1)$ to all sequences of sums (1).

Proposition 1. *For any fixed natural numbers p and q $(>p)$, the sequence $\sigma_n(p, q)$ is strictly increasing, in symbols:*

$$\sigma_n(p, q) \uparrow \quad (n=1, 2, \ldots). \tag{5}$$

Therefore,

$$\underline{\mathcal{B}}(p, q) \overset{\text{def}}{=} \sigma_1(p, q) \leqslant \sigma_n(p, q) < \ln \frac{q}{p} \overset{\text{def}}{=} \overline{\mathcal{B}}(p, q)$$

$$(p<q;\ p,\ q \text{ natural numbers};\ n=1, 2, \ldots),$$

and these bounds are the best possible for any two fixed natural numbers p and q $(>p)$.

Proof. For $p<q$, we have

$$\sigma_n(p, q) = \sum_{s=p}^{q-1} \sigma_n(s, s+1).$$

Hence, to prove (5), it suffices to prove that for $p=1, 2, \ldots$

$$\sigma_n(p, p+1) \uparrow \quad (n=1, 2, \ldots), \tag{6}$$

i.e.

$$a_n(p) \overset{\text{def}}{=} \sigma_{n+1}(p, p+1) - \sigma_n(p, p+1) > 0 \qquad (n, p=1, 2, \ldots).$$

Since

$$\frac{n+1}{n} \cdot \frac{1}{pn+p+k} - \frac{1}{pn+k} = \frac{k}{n(pn+p+k)(pn+k)} > 0,$$

or

$$\frac{1}{pn+k} < \frac{n+1}{n} \frac{1}{pn+p+k},$$

we get

$$a_n(p)=\sum_{k=p(n+1)+1}^{(p+1)(n+1)}\frac{1}{k}-\sum_{k=pn+1}^{(p+1)n}\frac{1}{k}=\frac{1}{p(n+1)+1}-\left(\sum_{k=pn+1}^{(p+1)n}\frac{1}{k}-\sum_{k=p(n+1)+2}^{(p+1)(n+1)}\frac{1}{k}\right)$$

$$=\frac{1}{pn+p+1}-\left(\sum_{k=1}^{n}\frac{1}{pn+k}-\sum_{k=1}^{n}\frac{1}{pn+p+k+1}\right)$$

$$=\frac{1}{pn+p+1}-\sum_{k=1}^{n}\frac{p+1}{(pn+k)(pn+p+k+1)}$$

$$>\frac{1}{pn+p+1}-\frac{n+1}{n}\sum_{k=1}^{n}\frac{p+1}{(pn+p+k)(pn+p+k+1)}$$

$$=\frac{1}{pn+p+1}-\frac{(n+1)(p+1)}{n}\sum_{k=1}^{n}\left(\frac{1}{pn+p+k}-\frac{1}{pn+p+k+1}\right)$$

$$=\frac{1}{pn+p+1}-\frac{(n+1)(p+1)}{n}\cdot\frac{n}{(pn+p+1)(pn+p+n+1)}=0$$

for $p, n=1, 2, \ldots$.

The second part of the proposition follows immediately from the first one and from the fact that

$$\lim_{n\to\infty}\sigma_n(p, q)=\ln\frac{q}{p}.$$

Proposition 2. *Let p and q $(>p)$ be two fixed natural numbers.*

1° *If $q\leqslant\frac{5}{2}p$, not being $p=2\alpha+1$, $q=5\alpha+\beta$ $(\alpha=2, 3, \ldots;\ \beta=1, 2)$ $\left(\text{especially, if } q<\frac{11}{5}p\right)$, then*

$$S_n(p, q)\ \downarrow\ (n=1, 2, \ldots).$$

In this case

$$\underline{B}(p, q)=\ln\frac{q}{p}<S_n(p, q)\leqslant S_1(p, q)=\overline{B}(p, q)\qquad(n=1, 2, \ldots),$$

with the best possible bounds.

1° *If $q\geqslant 3p$, then*

$$S_n(p, q)\ \uparrow\ (n=1, 2, \ldots).$$

In this case

$$\underline{B}(p, q)=S_1(p, q)\leqslant S_n(p, q)<\ln\frac{q}{p}=\overline{B}(p, q)\qquad(n=1, 2, \ldots),$$

with the best possible bounds again.

3° *For all other values of p and q $(>p)$, the sequence $S_n(p, q)$ is strictly decreasing for n large enough.*

Proof. To prove the parts of 1° and 2° concerning monotony, we use the following

Lemma. *For each fixed natural number n, the expression*

$$\beta_n(p, q) \overset{\text{def}}{=} S_{n+1}(p, q) - S_n(p, q)$$

is a strictly increasing function of q and a strictly decreasing function of p.

Indeed, it is easy to see that

$$\beta_n(p, q) = \sum_{k=qn+2}^{q(n+1)+1} \frac{1}{k} - \sum_{k=pn+1}^{p(n+1)} \frac{1}{k} = \gamma_n(q) + \delta_n(p) \tag{7}$$

and

$$\gamma_n(q) = \sum_{k=qn+1}^{q(n+1)} \frac{1}{k} + \frac{1}{q(n+1)+1} - \frac{1}{qn+1} = \sigma_q(n, n+1) + \frac{1}{q(n+1)+1} - \frac{1}{qn+1}.$$

So, in virtue of (6) and using

$$\left[\frac{1}{x(n+1)+1} - \frac{1}{xn+1}\right]' = \frac{n}{(xn+1)^2} - \frac{n+1}{[x(n+1)+1]^2}$$

$$= \frac{n(n+1)x^2 - 1}{(xn+1)^2[x(n+1)^2+1]^2} > 0 \qquad (x \geqslant 1),$$

we find that $\gamma_n(q)$ is a strictly increasing function of q. On the other hand, and again by (6),

$$\delta_n(p) = -\sum_{k=pn+1}^{p(n+1)} \frac{1}{k} = -\sigma_p(n, n+1)$$

is a strictly decreasing function of p, whict proves Lemma.

Now, we continue with the proof of Proposition 2. By (7),

$$\beta_n(1, 3) = \frac{1}{3n+2} + \frac{1}{3n+3} + \frac{1}{3n+4} - \frac{1}{n+1} \tag{8}$$

$$= \frac{6(n+1)}{(3n+2)(3n+4)} - \frac{2}{3(n+1)} = \frac{2}{3} \cdot \frac{9(n+1)^2 - (3n+2)(3n+4)}{(3n+2)(3n+4)(n+1)}$$

$$= \frac{2}{3} \cdot \frac{1}{(3n+2)(3n+4)(n+1)} > 0 \qquad (n = 1, 2, \ldots).$$

After some calculation, we find

$$\beta_n(2, 5) = -\frac{625n^4 + 1625n^2 + 3550n^2 + 604n + 72}{10(5n+1)(5n+2)(5n+3)(5n+4)(5n+6)(n+1)} < 0 \tag{9}$$

$$(n = 1, 2, \ldots).$$

By (8) and (9),

$$S_n(1, 3) \uparrow, \quad S_n(2, 5) \downarrow \quad (n = 1, 2, \ldots)$$

and this implies, for each $p = 1, 2, \ldots$,

$$S_n(p, 3p) = S_{np}(1, 3) \uparrow, \quad S_n(2p, 5p) = S_{np}(2, 5) \downarrow \quad (n = 1, 2, \ldots). \tag{10}$$

By (10) and Lemma, for $q \geqslant 3p$ we have

$$\beta_n(p, q) \geqslant \beta_n(p, 3p) > 0 \qquad (n=1, 2, \ldots).$$

Let $p < q \leqslant \frac{5}{2}p$. If $p=1$, q must be equal to 2, and the first part of 1° is valid in this case, being

$$\beta_n(1, 2) = \frac{1}{2n+2} + \frac{1}{2n+3} - \frac{1}{n+1} = \frac{1}{2n+3} - \frac{1}{2n+2} < 0 \qquad (n=1, 2, \ldots).$$

If $p=2a$ (a natural number), then $q \leqslant \frac{5}{2} \cdot 2a = 5a$, and by Lemma and (10),

$$\beta_n(p, q) \leqslant \beta_n(2a, 5a) < 0 \qquad (n=1, 2, \ldots).$$

If $p=2a+1$ (a natural number) and $q \neq 5a+1$, $q \neq 5a+2$, then the inequality $q \geqslant 5a+3$ is impossible, for it would imply

$$q = \left(q - \frac{5}{2}p\right) + \frac{5}{2}p \geqslant \left[5a+3 - \frac{5}{2}(2a+1)\right] + \frac{5}{2}p = \frac{1}{2} + \frac{5}{2}p > \frac{5}{2}p;$$

hence, $q \leqslant 5a$, i.e. by Lemma,

$$\beta_n(p, q) \leqslant \beta_n(2a+1, 5a) < \beta_n(2a, 5a) < 0 \qquad (n=1, 2, \ldots).$$

Finally, if $p=2 \cdot 1+1=3$ and $q=5 \cdot 1+2=7$, then, following a somewhet longer calculation, the expression $\beta_n(p, q)$ can be reduced to the form of a rational function of n having all coefficients of numerator negative and all those of denominator positive, which means that

$$\beta_n(3, 6) \leqslant \beta_n(3, 7) < 0 \qquad (n=1, 2, \ldots).$$

So, we have proved both first statements under 1° and 2°.

The last two statements in 1° and 2° follow from the first ones and the fact that for $p < q$

$$\lim_{n \to \infty} S_n(p, q) = \ln \frac{q}{p}.$$

Let the numbers p and q, with $p < q < 3p$, be fixed. For $n \geqslant 2$, we have

$$\begin{aligned}
\beta_n(p, q) &= \sum_{k=2}^{q+1} \frac{1}{qn+k} - \sum_{k=1}^{p} \frac{1}{pn+k} \\
&= \sum_{k=2}^{q+1} \frac{1}{qn}\left(1 + \frac{k}{qn}\right)^{-1} - \sum_{k=1}^{p} \frac{1}{pn}\left(1 + \frac{k}{pn}\right)^{-1} \\
&= \sum_{k=2}^{q+1} \frac{1}{qn}\left[1 - \frac{k}{qn} + O\left(\frac{1}{n^2}\right)\right] - \sum_{k=1}^{p} \frac{1}{pn}\left[1 - \frac{k}{pn} + O\left(\frac{1}{n^2}\right)\right] \\
&= \left(q \cdot \frac{1}{q} - p \cdot \frac{1}{p}\right)\frac{1}{n} + \left(\frac{1}{p^2}\sum_{k=1}^{p} k - \frac{1}{q^2}\sum_{k=1}^{q+1} k\right)\frac{1}{n^2} + O\left(\frac{1}{n^3}\right) \\
&= \frac{q-3p}{2pq} \cdot \frac{1}{n^2} + O\left(\frac{1}{n^3}\right) < 0
\end{aligned}$$

or n large enough. This proves 3°.

2. One of the advantages of the bounds

$$\underline{\mathcal{B}}(p, q), \quad \overline{\mathcal{B}}(p, q), \quad \underline{B}(p, q) \quad \text{and} \quad \overline{B}(p, q),$$

that we have obtained in this paper, is that they do not depend on n and that they are the best possible in the sense given in **0**, although for some values of p, q and n, one of both of the corresponding bounds of LUPAS **[2]** $\underline{\mathcal{L}}_n(p, q)$, $\overline{\mathcal{L}}_n(p, q)$, $\underline{L}_n(p, q)$ and $\overline{L}_n(p, q)$ might be sharper, and ours being sharper for other values of p, q and n. Namely, we have

Proposition 3. *With already introduced symbols:*

1° *For n large enough:*

$$\overline{\mathcal{B}}(p, q)<\overline{\mathcal{L}}_n(p, q), \text{ if } p<q;$$

$$\overline{B}(p, q)<\overline{L}_n(p, q), \text{ if } q \geqslant 3p;$$

$$\underline{L}_n(p, q)<\underline{B}(p, q), \text{ if } p<q \leqslant \frac{5}{2} p,$$

and not being

$$p=2\alpha+1, \quad q=5\alpha+\beta \qquad (\alpha=2, 3, \ldots; \ \beta=1, 2).$$

Here the lower bound for n depends on p and q.

2° $\underline{L}_n(p, 3p)<\underline{B}(p, 3p)$, $\overline{B}(p, 3p)<\overline{L}_n(p, 3p)$ $(p, n=1, 2, \ldots)$.

3° *There are values for p, q and n for which*

$$\underline{\mathcal{B}}(p, q)<\underline{\mathcal{L}}_n(p, q), \ \overline{\mathcal{B}}(p, q)>\overline{\mathcal{L}}_n(p, q), \ \underline{B}(p, q)<\underline{L}_n(p, q),$$

$$\overline{B}(p, q)>\overline{L}_n(p, q).$$

Namely:

3°.1. $\underline{\mathcal{B}}(p, q)<\underline{\mathcal{L}}_n(p, q)$ *for n large enough, if* $p<q \leqslant p+2$ *(the lower bound for n depends on p and q);*

3°.2. $\overline{\mathcal{B}}(1, 2)>\overline{\mathcal{L}}_n(1, 2)$ $(n=1, 2, 3)$, $\overline{\mathcal{B}}(1, 3)>\overline{\mathcal{L}}_n(1, 3)$ $(n=1, 2)$;

3°.3. $\underline{B}(3, 5)<\underline{L}_1(3, 5)$;

3°.4. *if m* $(=1, 2, \ldots)$ *is fixed, then for p large enough*

$$\overline{B}(p, p+m)>\overline{L}_n(p, p+m),$$

provided n is large enough (the lower bound for p depends on m and the lower bound for n on m and p); in special cases:

$\overline{B}(p, p+1)>\overline{L}_n(p, p+1)$ *for each p with n large enough,*

$\overline{B}(p, p+2)>\overline{L}_n(p, p+2)$ *for* $p=2, 3, \ldots$ *with n large enough.*

Proof. According to the LUPAS' estimate (3), the corresponding bounds for the sum $\sigma_n(p, q)$ are

$$(11) \qquad \underline{\mathcal{L}}_n(p, q)=2\frac{(q-p)\,n}{(q+p)\,n+1} \quad \text{and} \quad \overline{\mathcal{L}}_n(p, q)=\frac{(q-p)\left[(q+p)\,n+1\right]}{2q\,(pn+1)}.$$

By (2) and (11), for $p<q$ we have

(12)
$$\lim_{n\to\infty} \overline{\mathscr{L}}_n(p, q) = \lim_{n\to\infty} \overline{L}_n(p, q) = \frac{q^2-p^2}{2qp} = \frac{1}{2}\left[\frac{q}{p} - \left(\frac{q}{p}\right)^{-1}\right],$$
$$\lim_{n\to\infty} \underline{L}_n(p, q) = 2\cdot\frac{q-p}{q+p} = 2\cdot\frac{\frac{q}{p}-1}{\frac{q}{p}+1}.$$

Statement 1°, according to the corresponding parts of Propositions 1 and 2, follows from (12) and this double inequality

$$2\,\frac{t-1}{t+1} < \ln t < \frac{1}{2}\left(t - \frac{1}{t}\right) \quad (t>1),$$

which can easily be proved in the usual manner.

Further, for p, $n=1, 2, \ldots$, we have:

$$\underline{L}_n(p, 3p) = 2\cdot\frac{2pn+1}{4pn+2} = 1 < \frac{13}{12} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$
$$= S_1(1, 3) \leqslant S_p(1, 3) = S_1(p, 3p) = \underline{B}(p, 3p);$$
$$\overline{B}(p, 3p) = \ln 3 < \frac{9}{8} = \overline{L}_1(1, 3) \leqslant \overline{L}_{pn}(1, 3) = \overline{L}_n(p, 3p),$$

since:

$$e^{\frac{9}{8}} = e\cdot e^{\frac{1}{8}} > 2{,}7\cdot\left(1+\frac{1}{8}\right) = \frac{2{,}7\cdot 9}{8} = \frac{24{,}3}{8} > \frac{24}{8} = 3 \Rightarrow \ln 3 < \frac{9}{8};$$

$$\overline{L}_n(p, 3p) = \overline{L}_{pn}(1, 3) = \frac{(2pn+1)^2}{(pn+1)(3pn+1)},$$

$$\left[\frac{(2x+1)^2}{(x+1)(3x+1)}\right]' = \frac{2x(2x+1)}{(x+1)^2(3x+1)^2} > 0 \qquad (x>0).$$

So we have proved statement 2°.

From $p<q\leqq p+2$ it follows

$$\frac{1}{p+1} \leqq \frac{2}{q+p}$$

and so, by Proposition 1 and (11),

$$\underline{\mathscr{B}}(p, q) = \sigma_1(p, q) = \frac{1}{p+1} + \cdots + \frac{1}{q} \gtreqless \frac{q-p}{p+1}$$
$$\gtreqless \frac{2(q-p)}{q+p} = \lim_{n\to\infty} \underline{\mathscr{L}}_n(p, q),$$

which proves 3°.1.

Since

$$e^{\frac{5}{8}} < 1+\frac{5}{8}+\frac{1}{2}\cdot\left(\frac{5}{8}\right)^2+\frac{1}{6}\cdot\left(\frac{5}{8}\right)^3\cdot 3=\frac{13}{8}+\frac{25}{128}+\frac{125}{1024}$$

$$<\frac{13}{8}+\frac{1}{5}+\frac{1}{8}=\frac{7}{4}+\frac{1}{5}<2 \Rightarrow \ln 2>\frac{5}{8},$$

$$\overline{\mathcal{L}}_n(1,2)=\frac{3n+1}{4(n+1)} \uparrow, \ \overline{\mathcal{L}}_n(1,3)=\frac{4n+1}{3(n+1)} \uparrow \quad (n=1, 2, \ldots),$$

we get

$$\overline{\mathcal{B}}(1,2)=\ln 2>\frac{5}{8}=\overline{\mathcal{L}}_3(1,2)\geqslant\overline{\mathcal{L}}_n(1,2) \qquad (n=1, 2, 3),$$

$$\overline{\mathcal{B}}(1,3)=\ln 3>1=\overline{\mathcal{L}}_2(1,3)\geqslant\overline{\mathcal{L}}_n(1,3) \qquad (n=1, 2)$$

and inequalities 3°.2 are so proved.

3°. 3. follows from:

$$\underline{B}(3,5)=\ln\frac{5}{3}, \quad \underline{L}_1(3,5)=\frac{3}{5},$$

$$e^{\frac{3}{5}}>1+\frac{3}{5}+\frac{1}{2}\cdot\frac{9}{25}=\frac{89}{50}>\frac{5}{3} \Rightarrow \ln\frac{5}{3}<\frac{3}{5}.$$

If $m\,(=1, 2, \ldots)$ is fixed, then for $p\geqslant m$

$$p+m=\left(1+\frac{m}{p}\right)p\leqslant 2p$$

and so (Proposition 2 and (2))

$$\overline{B}(p,p+m)=S_1(p,p+m)=\frac{1}{p+1}+\cdots+\frac{1}{p+m+1}$$

$$>\frac{m+1}{p+m+1}=\frac{m+1}{p+m+1}-\frac{m(2p+m)}{2p(p+m)}+\frac{m(2p+m)}{2p(p+m)}$$

$$=\frac{2p^2-pm^2-m^2(m+1)}{2p(p+m+1)(m+p)}+\frac{m(2p+m)}{2p(p+m)}>\frac{m(2p+m)}{2p(p+m)}$$

$$=\lim_{p\to\infty}\overline{L}_n(p,p+m)$$

for p large enough. This immediately implies the general statement in 3°.4. In special cases:

$$\overline{B}(p,p+1)=\frac{1}{p+1}+\frac{1}{p+2}=\frac{2p+3}{(p+1)(p+2)}-\frac{2p+1}{2p(p+1)}+\frac{2p+1}{2p(p+1)}$$

$$=\frac{2(p^2-1)+p}{2p(p+1)(p+2)}+\lim_{n\to\infty}L_n(p,p+1)$$

$$>\lim_{n\to\infty}\overline{L}_n(p,p+1) \qquad (p=1, 2, \ldots),$$

$$\overline{B}(p, p+2)=\frac{1}{p+1}+\frac{1}{p+2}+\frac{1}{p+3}-\frac{2(p+1)}{p(p+2)}+\frac{2(p+1)}{p(p+2)}$$

$$=\frac{p^2-3}{p(p+1)(p+2)(p+3)}+\lim_{n\to\infty}\overline{L}_n(p, p+2)$$

$$>\lim_{n\to\infty}\overline{L}_n(p, p+2) \qquad (p=2, 3, \ldots),$$

which proves last two statements in 3°. 4.

3. Some remarks

3.1. Proposition 2 does not solve the problem of monotony and the best possible bounds of the sequence $S_n(p, q)$ $(n=1, 2, \ldots)$ for arbitrary fixed p and q, since the cases when

$$\text{(13)} \qquad \frac{5}{2}p<q<3p, \quad \text{or} \quad p=2\alpha+1,\ q=5\alpha+\beta \qquad (\alpha=2, 3, \ldots;\ \beta=1, 2)$$

remain open.

Nevertheless, in virtue of statement 3° of Proposition 2, as well as in virtue of some examined special cases, *we conjecture that also in the cases* (13) *sequence* (1) *is strictly decreasing and so has the best possible bounds*

$$\underline{B}(p, q)=\ln\frac{q}{p} \quad \textit{and} \quad \overline{B}(p, q)=S_1(p, q).$$

3.2. The bounds of $\sigma_n(p, q)$ and $S_n(p, q)$ $(p<q;\ n=1, 2, \ldots)$ which do not depend on n and which do not involve complicated expressions, but not being, of course, all the best possible, can be determined in the following way. For $p<q$, $n=1, 2, \ldots$:

$$\sigma_n(p, q)=\sum_{k=pn+1}^{qn}\frac{1}{k}>\int_{pn+1}^{qn+1}\frac{dt}{t}=\ln\frac{qn+1}{pn+1}=\ln\left[\frac{q}{p}-\frac{q-p}{p(pn+1)}\right]\geqslant\ln\frac{q+1}{p+1};$$

$$\sigma_n(p, q)<\int_{pn}^{qn}\frac{dt}{t}=\ln\frac{q}{p};$$

$$S_n(p, q)=\sum_{k=pn+1}^{qn+1}\frac{1}{k}>\int_{pn+1}^{qn+2}\frac{dt}{t}=\ln\frac{qn+2}{pn+1}$$

$$=\ln\left[\frac{q}{p}+\frac{2p-q}{p(pn+1)}\right]\begin{cases}\geqslant\ln\dfrac{q}{p} & (q\leqslant 2p)\\[2ex] \geqslant\ln\dfrac{q+2}{p+1} & (q>2p);\end{cases}$$

$$S_n(p, q)<\int_{pn}^{qn+1}\frac{dt}{t}=\ln\frac{qn+1}{pn}=\ln\left(\frac{q}{p}+\frac{1}{pn}\right)\leqslant\ln\frac{q+1}{p}.$$

Therefore, for $p<q$ and $n=1, 2, \dots,$

$$\ln\frac{q+1}{p+1}<\sigma_n(p, q)<\ln\frac{q}{p}; \tag{14}$$

$$\min\left\{\ln\frac{q}{p},\ \ln\frac{q+2}{p+1}\right\}<S_n(p, q)<\ln\frac{q+1}{p}. \tag{15}$$

Of all these bounds, the best possible ones are the upper bound in (14) and the lower bound in (15) when $q\leqslant 2p$.

Note that from (14) and (15), and according to

$$\lim_{n\to\infty}\sigma_n(p, q)=\lim_{n\to\infty}S_n(p, q)=\ln\frac{q}{p}\qquad(p<q),$$

it follows that $\ln\frac{q}{p}$ is the best possible lower bound of the sequence $\sigma_n(p, q)$ for $p<q$ and of the sequence $S_n(p, q)$ for $p<q\leqslant 2p$. This does not depend on the results previously obtained.

3.3. Finally, we remark that S. Barnard and J. M. Child in [3] have given the bounds $\frac{1}{2}$ and $\frac{2}{3}$ for $S_n(3, 5)$ $(n=1, 2, \dots)$. Both of them are weaker than the bounds given in Proposition 2, which follows from the fact that the bounds found in this paper are the best possible, and this can be directly verified:

$$e<2{,}77<\frac{25}{9}=\left(\frac{5}{3}\right)^2\Rightarrow\frac{1}{2}<\ln\frac{5}{3}=\underline{B}(3, 5),$$

$$\overline{B}(3, 5)=S_1(3, 5)=\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\frac{37}{60}<\frac{40}{60}=\frac{2}{3}.$$

REFERENCES

[1] *Problèm* 108, Matematički Vesnik **4** (19) (1967), 338.

[2] *Problèmes résolus*: 108, Matematički Vesnik **5** (20) (1968), 249.

[3] S. Barnand and J. M. Child, *Higher Algebra*, London 1955, p. 562.