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248.

A CYCLIC INEQUALITY*

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This is an expository article concerned with the cyclic inequality

$$\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\cdots+\frac{x_{n-1}}{x_n+x_1}+\frac{x_n}{x_1+x_2}\geqslant\frac{n}{2}, \tag{1}$$

where

$$x_i\geqslant 0,\ x_i+x_{i+1}>0 \qquad (x_{n+1}=x_1,\ x_{n+2}=x_2,\ i=1,\ldots,n) \tag{2}$$

and its generalizations.

The challenge of inequality (1) has created lively interest among mathematicians, as may be seen from the historical sketch which follows.

As will be seen, at the present time, the only undecided cases of (1) are $n=9$, 10, 11, 12, 13, 15, 17, 19, 21, 23, 25.

One can hope that this review of inequality (1) and allied inequalities will initiate some new contributions and in particular help to solve the mentioned undecided cases of (1).

The simplest case for $n=3$ of this inequality appeared in the literature [**1**] in 1903; though it is possible that it is not the first appearance.

In 1954 H. S. SHAPIRO [**2**] raised the question of proving (1) for all $n=3, 4, \ldots$. In 1956 a partial answer [**3**] to this question was obtained. Namely, the editors of The *American Mathematical Monthly* noted that M. J. LIGHTHILL succeeded in proving that (1) is not true for $n=20$. The editors also announced that H. S. SHAPIRO submitted the proof of (1) for $n=3$ and 4, and C. R. PHELPS for $n=5$. The counterexample of M. J. LIGHTHILL was published in detail in [**4**].

In 1958 L. J. MORDELL [**5**] has proved that (1) is true for $n=3$, 4, 5, 6. There is a short note [**6**] by A. ZULAUF appended to this paper by MORDELL. In this note he gave a counterexample by which he proved that (1) does not hold for $n=14$. This result implies that (1) is not true for even $n\geqslant 14$. Indeed, if we put

$$f_n(x_1,\ldots,x_n)=\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\cdots+\frac{x_{n-1}}{x_n+x_1}+\frac{x_n}{x_1+x_2},$$

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then the following funcitional identity holds

$$(3) \qquad f_{n+2}(x_1, \ldots, x_{n-1}, x_n, x_{n-1}, x_n) = f_n(x_1, \ldots, x_n) + 1.$$

Hence, if (1) is false for some n, then it is also false for $n+2$.

Another counterexample for $n=14$ was given later, 1960, by M. HERSCHORN and J. E. L. PECK [7].

All these counterexamples are of the same kind. Namely, if we set $n=2m$, $x_{2k}=x_k e$, $x_{2k-1}=1+b_k e$ $(k=1, \ldots, m)$, and e is sufficiently small and positive, then

$$f_{2m}(x_1, \ldots, x_{2m}) = m + qe^2 + O(e^3) \qquad (e \to 0),$$

where q is a quadratic form in a_k and b_k. If $m=7$ then $a_k \geqslant 0$ and b_k can be found such that $q<0$. For instance, in the counterexample by ZULAUF we have

$(a_1, \ldots, a_7) = (7, 6, 5, 2, 0, 1, 4)$, $(b_1, \ldots, b_7) = (7, 4, 1, 0, 1, 4, 6)$ and $q=-2$.

DINA GLADYS S. THOMAS [8] has proved that, for $n=8$, 10, 12, q is a positive definite quadratic form in a_k and b_k. Hence, in these cases there does not exist a counterexample of this kind.

Let us define

$$\mu(n) = \inf_{x_i \geqslant 0} f_n(x_1, \ldots, x_n), \quad \lambda(n) = \frac{\mu(n)}{n}.$$

Identity (3) implies that

$$(4) \qquad \mu(n+2) \leqslant \mu(n) + 1.$$

In 1958 R. A. RANKIN [9] has proved that there exists $\lambda = \lim_{n \to +\infty} \lambda(n)$ and

$$(5) \qquad \lim_{n \to +\infty} \lambda(n) = \inf_{n \geqslant 1} \lambda(n).$$

He proved also that $\lambda < \frac{1}{2} - 7 \cdot 10^{-8}$, which implies that (1) is false for sufficiently large n. In his later paper [10] he proved that $\lambda > 0{,}33$.

The upper bound for λ was improved by A. ZULAUF [11]. He proved that $\lambda < 0{,}49950317$. In the same note he proved that (1) does not hold for $n=53$, i.e., $\lambda(53) < 1/2$. This and (4) imply that $\lambda(n) < 1/2$ for odd $n \geqslant 53$.

L. J. MORDELL [12] has proved that (1) holds for $n=7$ if besides (2) we have

$$(6) \qquad x_1 \geqslant x_7 \geqslant x_2 \geqslant x_6, \quad x_4 \geqslant x_2, \quad x_1 \geqslant x_3.$$

K. GOLDBERG informed L. J. MORDELL in his letter of February 9, 1960 that he has checked (1) for $n=7$ and he found it to be true for 300.000 pseudo-random values of x.

A. ZULAUF [13] remarked that the supplementary inequalities (6) are very restrictive. He proved that if n positive random numbres are chosen, then the probability that (1) is satisfied is at least 1/2. This is implied by the inequality

$$f_n(x_1, \ldots, x_n) + f_n(x_n, x_{n-1}, \ldots, x_1) \geqslant n.$$

The proof of the last inequality follows:

If $A_k = x_k + x_{k+1}$ $(x_{k+n} = x_k)$, then

$$f_n(x_1, \ldots, x_n) + f_n(x_n, x_{n-1}, \ldots, x_1)$$
$$= \sum_{k=1}^{n} \frac{x_k + x_{k+3}}{A_{k+1}} = \sum_{k=1}^{n} \frac{A_k - A_{k+1} + A_{k+2}}{A_{k+1}}$$
$$= -n + \sum_{k=1}^{n} \frac{A_k}{A_{k+1}} + \sum_{k=1}^{n} \frac{A_{k+2}}{A_{k+1}} \geqslant n,$$

since

$$\sum_{k=1}^{n} \frac{A_k}{A_{k+1}} \geqslant n, \quad \sum_{k=1}^{n} \frac{A_{k+2}}{A_{k+1}} \geqslant n.$$

P. H. Dianánda **[14]** has obtained the following result: If (2) is satisfied and $x_{m+1}, \ldots, x_{m+n}$ $(x_{k+n} = x_k)$ is monotone for some natural number m, then (1) is true.

The bounds for λ were improved by P. H. Dianánda in **[15]** and **[16]**. Namely, he proved that

$$0{,}461238 < \lambda < 0{,}499197.$$

In 1963, D. Ž. Đoković **[17]** proved that (1) is true for $n = 8$. P. H. Dianánda **[18]** and B. Bajšanski **[19]** have proved independently of each other that this result of Djoković implies that (1) is true for $n = 7$. More generally, they proved that

$$\mu(2m) \leqslant \mu(2m-1) + \frac{1}{2}. \tag{7}$$

In the same paper, P. H. Dianánda has proved that (1) is not true for $n = 27$. So, now the only undecided cases of (1) are $n = 9$, 10, 11, 12, 13, 15, 17, 19, 21, 23, 25.

A. Zulauf **[20]** has proved that for the modified cyclic sum

$$\sum x_k/(x_k + x_{k+1})$$

the following inequality holds

$$1 < \frac{x_1}{x_1 + x_2} + \frac{x_2}{x_2 + x_3} + \cdots + \frac{x_{n-1}}{x_{n-1} + x_n} + \frac{x_n}{x_n + x_1} < n - 1,$$

where $x_1, \ldots, x_n$ $(n \geqslant 3)$ are nonnegative, and all denominators positive. These bounds are the best possible.

Let $x_1, \ldots, x_7$ be any nonnegative numbers such that $A_k = x_k + x_{k+1}$ $(x_{k+7} = x_k)$ is positive for $k = 1, \ldots, 7$. In this case A. Zulauf **[21]** has proved the following inequalities

$$\sum \frac{x_k}{A_{k+2}} \geqslant 3, \quad \sum \frac{x_k}{A_{k+4}} \geqslant 3, \quad \sum \frac{x_k}{A_{k+3}} \geqslant 2 \tag{8}$$

and some others. All bounds in (8) are the best possible. For instance, putting $(x_1, \ldots, x_7) = (2t^2, t, 0, t^2, t^2, 0, t)$, we obtain

$$\sum \frac{x_k}{A_{k+3}} = 1 + \frac{2}{t} + \frac{2t}{2t+1} \to 2 \qquad (t \to +\infty).$$

In [5] L. J. MORDELL conjectured that

$$\left(\sum_{i=1}^{n} x_i\right)^2 \geqslant k \sum_{i=1}^{n} x_i (x_{i+1} + \cdots + x_{i+m}), \tag{9}$$

where $x_{n+l} = x_l$, and $x_1, \ldots, x_n$ are all nonnegative, $n \geqslant m+1$ and

$$k = \min\left(\frac{n}{m}, \frac{2m+2}{m}\right).$$

He proved this assertion for $n = m+1$, $n = m+2$ and for $n \geqslant 2m$.

It was remarked by T. MURPHY that this conjecture does not hold for $m = 4$, $n = 7$. The best possible value of k is $12/7$, which is smaller that $7/4$.

In the same papar L. J. MORDELL proved that the minimum values F_n, G_n of

$$F_n(x) = \sum_{i=1}^{n} (x_i^2 - 2x_i x_{i+1}) \qquad (x_{n+1} = x_1),$$

$$G_n(x) = \sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n-1} x_i x_{i+1}$$

are

$$F_n = -\frac{1}{n} \quad (n = 3, 4, 5), \qquad F_n = -\frac{1}{6} \quad (n \geqslant 6),$$

$$G_2 = 0, \quad G_3 = -\frac{1}{7}, \quad G_n = -\frac{1}{6} \quad (n \geqslant 4),$$

where $x_i \geqslant 0$ $(i = 1, \ldots, n)$ and $x_1 + \cdots + x_n = 1$.

In [22] P. H. DIANANDA obtained interesting results concerning (9), namely: Let $k(m, n)$ be the largest value of k, such that (9) holds. Then

1° $k(m, n) = \frac{n}{m}$ if $n \mid m+2$ or $2m$ or $2m+1$ or $2m+2$,

or if $n \mid m+3$ and $n = 8$ or 9 or 12,

or if $n \mid m+4$ and $n = 12$,

$k(m, n) < \frac{n}{m}$ otherwise;

2° $k(m, n) = \frac{2m+2}{n}$ if $n > 2m+2$;

3° $k(m, n) = \frac{12n}{n+12m-6}$ if $n \mid 2m-1$ and $n > 6$;

4° $k(m, n) = \dfrac{k(m-rn, n)}{1+rk(m-rn, n)}$ if $rn<m$ $(r=1, 2, \ldots)$;

5° $k(m, n+1) \geqslant k(m, n) \geqslant k(m+1, n)$.

V. J. D. Baston [23] has complemented these results by the following:

6° $k(m, 2m-t) = \dfrac{4(t+2)}{3t+4}$ if $t \neq 3$ and $m \geqslant \max\left(2t, \dfrac{3t}{2}+2\right)$;

7° $k(7, 11) \leqslant k(m, 2m-3) \leqslant \dfrac{20}{13}$ for $m \geqslant 7$;

8° $k(m, n) \leqslant \dfrac{2(r+1)[(r+1)(m+1)-rn]}{(2r+1)[(r+1)m-rn]+2r(r+1)}$ if $m+2<n \leqslant 2m-1$

and r is an integer such that $\dfrac{r+2}{r+1} \leqslant \dfrac{n}{m} < \dfrac{r+1}{r}$.

P. H. Diananda [24] also considered the following two inequalities

(10) $$\sum_{i=1}^{n} \frac{x_i}{x_{i+1}+\cdots+x_{i+m}} \geqslant \frac{n}{m},$$

(11) $$\left(\sum_{i=1}^{n} x_i\right)^2 \geqslant \frac{n}{m} \sum_{i=1}^{n} x_i (x_{i+1}+\cdots+x_{i+m}),$$

where $x_{n+k}=x_k$ and $x_i>0$ $(i=1, \ldots, n)$, and proved the following results:

1° Inequality (10) is true if

(12) $$\sin\frac{r}{n}\pi \geqslant \sin(2m+1)\frac{r}{n}\pi \qquad \left(r=1, \ldots, \left[\frac{n}{2}\right]\right).$$

2° Inequality (10) is true if

(13) $$n \mid m+2, \text{ or } 2m, \text{ or } 2m+1, \text{ or } 2m+2.$$

3° If (12) is true, then inequality (11) holds.

4° If (13) is true, (12) is true.

In [25] P. H. Diananda obtained more general results, as for example,

$$\inf_{x, y, z>0} \left(\frac{x^2}{yz}+\frac{4y^2}{yz+zx}+\frac{(x+2z)^2}{zx+2xy}\right)=6.$$

Let x_i $(i=1, \ldots, n)$ be positive real numbres and $x_{n+k}=x_k$. P. H. Diananda in three papers [16], [26], [27] has proved respectively the following results:

$$\inf_{x_1, \ldots, x_n} \frac{4}{n} \sum_{i=1}^{n} \frac{x_i}{3x_{i+1}+x_{i+2}+|x_{i+1}-x_{i+2}|} = 2^s - s \qquad \left(s=\frac{1}{n}\left[\frac{n}{3}\right]\right),$$

$$\inf_{x_1, \ldots, x_n} \sum_{i=1}^{n} \frac{x_i^2}{x_{i+1}^2-x_{i+1}x_{i+2}+x_{i+2}^2} = \left[\frac{n+1}{2}\right],$$

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1}+\cdots+x_{i+m}} \geqslant \frac{1}{m}\left[\frac{n+m-1}{m}\right] \geqslant \frac{n}{m^2}.$$

Finally, we note the following cyclic inequality [28]

$$\frac{x_1+\cdots+x_k}{x_{k+1}+\cdots+x_n}+\frac{x_2+\cdots+x_{k+1}}{x_{k+2}+\cdots+x_1}+\cdots+\frac{x_n+x_1+\cdots+x_{k-1}}{x_k+\cdots+x_{n-1}}\geqslant\frac{nk}{n-k},$$

where $x_1, \ldots, x_n$ are positive numbers, and $n \geqslant k > 1$.

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