PUBLIKACIJE ELEKTROTEHNIČKOG FAKULTETA UNIVERZITETA U BEOGRADU
PUBLICATIONS DE LA FACULTÉ D'ÉLECTROTECHNIQUE DE L'UNIVERSITÉ À BELGRADE

SERIJA: MATEMATIKA I FIZIKA — SÉRIE: MATHÉMATIQUES ET PHYSIQUE

№ 247 — № 273 (1969)

247. THE STEFFENSEN INEQUALITY*

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This paper concerns a general inequality due to J. F. Steffensen [1], published in 1918. The Steffensen inequality does not appear in the work *Inequalities* by G. H. Hardy, J. Littlewood and G. Pólya (Cambridge — first edition 1934, second edition 1952), which assembled almost all important inequalities.

In the *Jahrbuch über die Fortschritte der Mathematik* Steffensen's paper [1] has not been reviewed, but G. Szegö quoted the Steffensen inequality in his review of papers [2] and [4] by Hayashi.

J. F. Steffensen returned to his inequality a number of times and gave some generalizations, but only in 1959 did his article [7] from 1947 attract the attention of R. Bellman, who in his paper [8] pointed out that Steffensen's inequality is very general and implies many other inequalities which were established during the period 1950—1959.

This paper gives a history of the Steffensen inequality and some critical analyses, connects numerous isolated results, completes some proofs, and corrects a number of mistakes found in articles related to this inequality. It is interesting to note that some of these results have been rediscovered a number of times.

One can hope that this review of Steffensen's inequality will also initiate some new contributions.

The author wishes to thank Dr Roy O. Davies, Dr D. C. B. Marsh, Dr M. Marjanović *and* Dr D. Ž. Djoković *for valuable discussions held and suggestions made during the preparation of parts of this paper.*

While collecting material for this article, the author encountered some difficulties. Kobenhavns Universitets Matematiske Institut in Denmark, through its secretary, Madame Marina de Versmann, *was of great help to the author, sending him the necessary copies of articles published in Denmark and other countries. It is a pleasure for the author to be able to thank the afore-mentioned Institute.*

We shall present the subject chronologically.

The following result and its proof were given by Steffensen [1].

Theorem 1. *Assume that two integrable functions $f(t)$ and $g(t)$ are defined on the interval (a, b), that $f(t)$ never increases and that $0 \leqslant g(t) \leqslant 1$ in (a, b). Then*

$$\int_{b-\lambda}^{b} f(t)\,dt \leqslant \int_{a}^{b} f(t)\,g(t)\,dt \leqslant \int_{a}^{a+\lambda} f(t)\,dt, \tag{1}$$

where

$$\lambda = \int_{a}^{b} g(t)\,dt. \tag{2}$$

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* Received January 20, 1969.

Proof. The second inequality of (1) may be derived as follows:

$$
\begin{aligned}
&\int_a^{a+\lambda} f(t)\,dt-\int_a^b f(t)\,g(t)\,dt\\
&\qquad=\int_a^{a+\lambda}[1-g(t)]f(t)\,dt-\int_{a+\lambda}^b f(t)\,g(t)\,dt\\
&\qquad\geqslant f(a+\lambda)\int_a^{a+\lambda}[1-g(t)]\,dt-\int_{a+\lambda}^b f(t)\,g(t)\,dt\\
&\qquad=f(a+\lambda)\Big[\lambda-\int_a^{a+\lambda} g(t)\,dt\Big]-\int_{a+\lambda}^b f(t)\,g(t)\,dt\\
&\qquad=f(a+\lambda)\Big[\int_a^b g(t)\,dt-\int_a^{a+\lambda} g(t)\,dt\Big]-\int_{a+\lambda}^b f(t)\,g(t)\,dt\\
&\qquad=f(a+\lambda)\int_{a+\lambda}^b g(t)\,dt-\int_{a+\lambda}^b f(t)\,g(t)\,dt\\
&\qquad=\int_{a+\lambda}^b g(t)\,[f(a+\lambda)-f(t)]\,dt\\
&\qquad\geqslant 0.
\end{aligned}
$$

The first inequality of (1) can be proved similarly. However, the second inequalty of (1) implies the first.

Indeed, let $G(t)=1-g(t)$ and $\Lambda=\int_a^b G(t)\,dt$. Note that $0\leqslant G(t)\leqslant 1$ if $0\leqslant g(t)\leqslant 1$ in (a, b). Then

(3) $$b-a=\lambda+\Lambda.$$

Suppose the second inequality of (1) holds. Then

$$\int_a^b f(t)\,G(t)\,dt\leqslant\int_a^{a+\Lambda} f(t)\,dt,$$

i.e.,

$$\int_a^b f(t)\,[1-g(t)]\,dt\leqslant\int_a^{b-\lambda} f(t)\,dt,$$

i.e.,

$$\int_a^b f(t)\,dt-\int_a^{b-\lambda} f(t)\,dt\leqslant\int_a^b f(t)\,g(t)\,dt,$$

i.e.,

$$\int_{b-\lambda}^b f(t)\,dt\leqslant\int_a^b f(t)\,g(t)\,dt,$$

which is the first inequality of (1).

T. HAYASHI, in [2], generalizes inequality (1) slightly by taking the condition

$$0 \leqslant g(t) \leqslant A \qquad (A \text{ a constant } > 0)$$

instead of $0 \leqslant g(t) \leqslant 1$, and proves that

$$A \int_{b-\lambda}^{b} f(t)\, dt \leqslant \int_{a}^{b} f(t)\, g(t)\, dt \leqslant A \int_{a}^{a+\lambda} f(t)\, dt,$$

where

$$\lambda = \frac{1}{A} \int_{a}^{b} g(t)\, dt.$$

Some slight generalizations of the STEFFENSEN inequality are given in MEIDELL [3].

In [4], J. F. STEFFENSEN uses the second inequality of (1) to derive a generalization of JENSEN's inequality for continuous, convex functions. He proves the following:

Theorem 2. *If $f(x)$ is a continuous, convex function and x_k $(k = 1, \ldots, n)$ never decrease, and if c_k $(k = 1, \ldots, n)$ satisfy the conditions*

$$0 \leqslant \sum_{k=\nu}^{n} c_k \leqslant \sum_{k=1}^{n} c_k \qquad (\nu = 1, \ldots, n), \text{ with } \sum_{k=1}^{n} c_k > 0,$$

then

$$f\left(\frac{\sum_{k=1}^{n} c_k x_k}{\sum_{k=1}^{n} c_k} \right) \leqslant \frac{\sum_{k=1}^{n} c_k f(x_k)}{\sum_{k=1}^{n} c_k}. \tag{4}$$

This inequality is evidently more general than JENSEN's inequality [5] since the numbers c_k $(k = 1, \ldots, n)$ need not necessarly be positive.

A corresponding inequality for integrals is given as well:

Theorem 3. *If $f(x)$ is a continuous, convex function, $g(x)$ never increases and $h(x)$ satisfies*

$$0 \leqslant \int_{\theta}^{1} h(x)\, dx \leqslant \int_{0}^{1} h(x)\, dx, \text{ with } 0 \leqslant \theta \leqslant 1, \text{ and } \int_{0}^{1} h(x)\, dx > 0,$$

then

$$f\left(\frac{\int_{0}^{1} h(x)\, g(x)\, dx}{\int_{0}^{1} h(x)\, dx} \right) \leqslant \frac{\int_{0}^{1} h(x) f(g(x))\, dx}{\int_{0}^{1} h(x)\, dx}. \tag{5}$$

T. HAYASHI in [6] obtains an upper bound for the right-hand sides in (4) and (5).

Assuming that $f(t) \to 0$, for $t \to +\infty$, and that $f(t)$ is integrable in $(0, +\infty)$, J. F. STEFFENSEN [7] applies (1) to deduce the following inequalities:

$$\sum_{k=1}^{+\infty} (-1)^k f(k\pi) < \int_0^{+\infty} f(t) \cos t \, dt < \sum_{k=0}^{+\infty} (-1)^k f(k\pi),$$

$$\sum_{k=0}^{+\infty} (-1)^k f\left(\left(k+\frac{1}{2}\right)\pi\right) < \int_0^{+\infty} f(t) \sin t \, dt < f(0) + \sum_{k=0}^{+\infty} (-1)^k f\left(\left(k+\frac{1}{2}\right)\pi\right).$$

J. F. STEFFENSEN also gives more precise inequalities in terms of

$$g(x) = \int_x^{x+1} f(t) \, dt.$$

It should be noticed that R. BELLMAN, in [8], refers to the STEFFENSEN's paper [7] from 1947, as a source of inequality (1) but not to paper [1] from 1918, nor [4] from 1919, though this inequality was published for the first time in 1918. This is probably the reason why R. BELLMAN does not mention Theorems 2 and 3, in his paper [8], or monograph [9], published in cooperation with E. F. BECKENBACH.

R. BELLMAN [8] gives the following proof of STEFFENSEN's inequality (1) requiring $f(t)$ to be nonnegative.

Assuming that there does not exist an interval on which $f(t) = 0$, define the function $u(s)$ by the equality

$$\int_a^s f(t) g(t) \, dt = \int_a^{u(s)} f(t) \, dt, \tag{6}$$

whence $u(a) = a$, and

$$\int_a^{s+h} f(t) g(t) \, dt = \int_a^{u(s+h)} f(t) \, dt \qquad (a < s+h < b). \tag{7}$$

Let $h > 0$. Then (6) and (7) yield

$$\int_s^{s+h} f(t) g(t) \, dt = \int_{u(s)}^{u(s+h)} f(t) \, dt.$$

This equality is valid only if $u(s+h) \geqslant u(s)$, i.e., if $u(s)$ is monotone increasing.

Since $0 \leqslant g(t) \leqslant 1$ and $f(t) \geqslant 0$ $(0 < t < b)$, we have

$$0 \leqslant \int_a^s f(t) g(t) \, dt \leqslant \int_a^s f(t) \, dt \qquad (a < s < b). \tag{8}$$

From (6) and (8) it follows that

$$\int_a^{u(s)} f(t) \, dt \leqslant \int_a^s f(t) \, dt,$$

whence $u(s) \leqslant s$.

Starting from (6) and (7), we obtain

$$\left|\int_a^{u(s+h)} f(t)\,dt - \int_a^{u(s)} f(t)\,dt\right| = |u(s+h)-u(s)|\cdot\mu$$

$$= \left|\int_a^{s+h} f(t)\,g(t)\,dt - \int_a^{s} f(t)\,g(t)\,dt\right| = \left|\int_s^{s+h} f(t)\,g(t)\,dt\right|$$

$$\leq |h|\,f(a),$$

where

$\inf f(t)\leq\mu\leq\sup f(t)$ for $t\in[s, s+h]$ if $h>0$ or $t\in[s+h, s]$ if $h<0$.

This proves the continuity of $u(s)$.

By differentiation equality (6) gives

$$f(u)\frac{du}{ds}=f(s)\,g(s) \quad \text{(almost everywhere)},$$

whence

$$\frac{du}{ds}=\frac{f(s)}{f(u)}g(s)\leq g(s),$$

taking account of the fact that $u(s)\leq s$ and that $f(s)$ is monotone decreasing. Hence

$$\int_a^s du \leq \int_a^s g(s)\,ds,$$

i.e.,

$$u(s)\leq a+\int_a^s g(s)\,ds. \tag{9}$$

(6) and (9) yield the right-hand inequality of (1).

Using the same procedure, R. Bellman, in [**8**], also establishes one of, as he points out, many possible generalizations of Steffensen's inequality. His generalization reads:

Theorem 4. *Let $f(t)$ be a nonnegative and monotone decreasing function in $[a, b]$ and $f\in L^p\,[a, b]$, and let $g(t)\geq 0$ in $[a, b]$ and $\int_a^b g(t)^q\,dt\leq 1$, where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\left(\int_a^b f(t)\,g(t)\,dt\right)^p \leq \int_a^{a+\lambda} f(t)^p\,dt,$$

where

$$\lambda=\left(\int_a^b g(t)\,dt\right)^p.$$

Proof. Consider the function $u(s)$ defined in $[a, b]$ by the equation

(10)
$$\Big(\int_a^s f(t)\,g(t)\,dt\Big)^p=\int_a^{u(s)} f(t)^p\,dt.$$

By HÖLDER's inequality, taking in to consideration the assumptions of Theorem 4, we have

$$\int_a^s f(t)\,g(t)\,dt\leqslant\Big(\int_a^s f(t)^p\,dt\Big)^{1/p}\Big(\int_a^s g(t)^q\Big)^{1/q}$$
$$\leqslant\Big(\int_a^s f(t)^p\,dt\Big)^{1/p},$$

i.e.,

(11)
$$\Big(\int_a^s f(t)\,g(t)\,dt\Big)^p\leqslant\int_a^s f(t)^p\,dt.$$

Starting from (10) and (11), we can prove that $u(s)$ exists and satisfies $u(s)\leqslant s$ in $[a, b]$, with $u(a)=a$. The function $u(s)$ is monotone increasing and satisfies the differential equation

$$f(u)^p\,\frac{du}{ds}=p f(s)\,g(s)\left(\int_a^s f(t)\,g(t)\,dt\right)^{p-1}$$

almost everywhere.

The monotonic nature of $f(s)$ and $u(s)$ yields the inequality

$$\frac{du}{ds}\leqslant p g(s)\left(\int_a^s g(t)\,dt\right)^{p-1},$$

whence

$$u(s)\leqslant a+\Big(\int_a^s g(t)\,dt\Big)^p$$

which completes the proof.

Remark. We shall prove that the restriction $f(t)\geqslant 0$ made in the above proof by BELLMAN of STEFFENSEN's inequality can be removed. Indeed, let $f(t)$ be an arbitrary decreasing function on $[a, b]$. Then the function F defined by $F(t)=f(t)-f(b)$ is nonnegative and decreasing. Now, following R. BELLMAN **[8]**, the STEFFENSEN inequality can be applied to $F(t)$, namely

$$\int_{b-\lambda}^{b}(f(t)-f(b))\,dt\leqslant\int_a^b(f(t)-f(b))\,g(t)\,dt$$
$$\leqslant\int_a^{a+\lambda}(f(t)-f(b))\,dt,$$

i.e.,

$$\int_{b-\lambda}^{b} f(t)\,dt \leqslant \int_{a}^{b} f(t)\,g(t)\,dt \leqslant \int_{a}^{a+\lambda} f(t)\,dt.$$

This proves the above assertion that the restriction $f(t)\geqslant 0$ in BELLMAN's proof is not essential.

BELLMAN's paper **[8]** of 1959 was preceded by a series of notes in which various inequalities, actually all of the STEFFENSEN type, are established. These were given in what follows.

G. SZEGÖ **[10]** proved in 1950 the following result:

Theorem 5. *If* $a_1 > a_2 > \cdots > a_{2m-1} > 0$ *and* $f(x)$ *is a continuous, convex function in* $[0, a_1]$, *then*

$$\sum_{k=1}^{2m-1} (-1)^{k-1} f(a_k) \geqslant f\left(\sum_{k=1}^{2m-1} (-1)^{k-1} a_k\right).$$

In 1952, H. F. WEINBERGER **[11]** proved Theorem 5 for the function $f(x)=x^r$ $(r>1)$, namely

Theorem 6. *If* $a_1 \geqslant \cdots \geqslant a_n \geqslant 0$, *then*

$$\sum_{k=1}^{n} (-1)^{k-1} a_k^r \geqslant \left(\sum_{k=1}^{n} (-1)^{k-1} a_k\right)^r \qquad (r>1).$$

In 1953, R. BELLMMN **[12]** proved a generalized version of Theorem 6:

Theorem 7. *Let* $a_1 \geqslant \cdots \geqslant a_n \geqslant 0$ *and let* $f(x)$ *be a continuous convex function in* $[0, a_1]$, *with* $f(0)\leqslant 0$. *Then*

$$\sum_{k=1}^{n} (-1)^{k-1} f(a_k) \geqslant f\left(\sum_{k=1}^{n} (-1)^{k-1} a_k\right).$$

We note that the condition $f(0)\leqslant 0$ cannot be relaxed if there is an even number of terms, but may be omitted if n is odd, as given in Theorem 5.

E. M. WRIGHT **[13]** in 1954 points out that Theorem 7 is a consequence of Theorem 108 in **[14]**, p. 89, wich reads:

Theorem 8. (*Majorization theorem*). *The conditions*

$$x_1 \geqslant \cdots \geqslant x_n, \quad y_1 \geqslant \cdots \geqslant y_n,$$

$$\sum_{i=1}^{k} x_i \leqslant \sum_{i=1}^{k} y_i \quad \textit{for} \quad k=1, \ldots, n-1,$$

and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

are necessary and sufficient in order that for every continuous, convex function f,

$$\sum_{i=1}^{n} f(x_i) \leqslant \sum_{i=1}^{n} f(y_i).$$

M. Biernacki [15] in 1954 proved:

Theorem 9. *If* $a_1 \geqslant \cdots \geqslant a_n > 0$, $b_1 \geqslant \cdots \geqslant b_n > 0, \ldots, h_1 \geqslant \cdots \geqslant h_n > 0$ *and* $p > 1$, *then*

$$(a_1^p + b_1^p + \cdots + h_1^p) - (a_2^p + b_2^p + \cdots + h_2^p) + \cdots \pm (a_n^p + b_n^p + \cdots + h_n^p)$$
$$\geqslant (a_1 - a_2 + \cdots \pm a_n)^p + (b_1 - b_2 + \cdots \pm b_n)^p + \cdots + (h_1 - h_2 + \cdots \pm h_n)^p.$$

The inequality is reversed for $0 < p < 1$.

This theorem is an immediate consequence of Theorem 6.

Theorem 10. *Let* $f(x)$ *be a continuous, convex function for* $x > 0$ *and let* $f(0) = 0$. *Further, let*

$$\begin{aligned} &0 < a_1 \leqslant b_1 \leqslant a_2,\\ &0 < a_2 \leqslant b_2 \leqslant a_3,\\ &\vdots\\ &0 < a_{n-1} \leqslant b_{n-1} \leqslant a_n,\\ &0 \leqslant b_n,\\ &b_1 + \cdots + b_n \leqslant a_1 + \cdots\cdot + a_k. \end{aligned}$$

Then

$$(f(a_1) + \cdots + f(a_n)) - (f(b_1) + \cdots + f(b_n))$$
$$\geqslant f((a_1 + \cdots + a_n) - (b_1 + \cdots + b_n)).$$

H. D. Brunk [16] proved in 1956 a general result having as a corollary

Theorem 11. *Let* $f(x)$ *be a continuous, convex function on* $[a, b]$, *with* $f(0) \leqslant 0$ *Let* $b \geqslant a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant 0$, *and let* $1 \geqslant h_1 \geqslant h_2 \geqslant \cdots \geqslant h_n \geqslant 0$. *Then*

$$\sum_{k=1}^{n} (-1)^{k-1} h_k f(a_k) \geqslant f\left(\sum_{k=1}^{n} (-1)^{k-1} h_k a_k\right). \tag{12}$$

If $h_1 = \cdots = h_n = 1$, *Theorem* 11 *reduces to Theorem* 7.

I. Olkin [17] in 1959 proved

Theorem 12. *Let* $1 \geqslant h_1 \geqslant \cdots \geqslant h_n \geqslant 0$ *and* $a_1 \geqslant \cdots \geqslant a_n \geqslant 0$. *Let* $F(x)$ *be* , *continuous, convex function on* $[0, a_1]$. *Then*

$$\left(1 - \sum_{k=1}^{n} (-1)^{k-1} h_k\right) F(0) + \sum_{k=1}^{n} (-1)^{k-1} h_k F(a_k) \geqslant F\left(\sum_{k=1}^{n} (-1)^{k-1} h_k a_k\right). \tag{13}$$

This theorem can be obtained from Theorem 11. Indeed, put $f(x) = F(x) - F(0)$. Then (12) becomes

$$\sum_{k=1}^{n} (-1)^{k-1} h_k (F(a_k) - F(0)) \geqslant F\left(\sum_{k=1}^{n} (-1)^{k-1} h_k a_k\right) - F(0),$$

which is precisely inequality (13).

Now, we shall prove that inequality (13) can be derived from the second inequality of (1). First, for $t \in [0, a_1]$, put

$$g(t) = \lambda_k \text{ for } a_{k+1} < t \leqslant a_k \ (k = 1, \dots, n), \text{ with } a_{n+1} = 0,$$

where

$$\lambda_1 = h_1, \quad \lambda_2 = h_1 - h_2, \dots, \lambda_n = h_1 - h_2 + \cdots + (-1)^{n-1} h_n,$$

whence $0 \leqslant g(t) \leqslant 1$ in $[0, a_1]$.

Since

$$\lambda = \int_0^{a_1} g(t)\,dt = (a_1 - a_2)\lambda_1 + (a_2 - a_3)\lambda_2 + \cdots + (a_{n-1} - a_n)\lambda_{n-1} + a_n \lambda_n$$
$$= a_1 h_1 - a_2 h_2 + \cdots + (-1)^{n-1} a_n h_n,$$

we have

$$\lambda = \sum_{k=1}^{n} (-1)^{k-1} h_k a_k .$$

Now, if $F(t)$ is a function with increasing first derivative $F'(t)$, then $f(t) = -F'(t)$ is a decreasing function.

The functions $g(t)$ and $f(t)$, defined above, satisfy all the conditions necessary and sufficient for application of STEFFENSEN's inequality

$$\int_a^b f(t) g(t)\,dt \leqslant \int_a^{a+\lambda} f(t)\,dt .$$

Thus we have

$$\int_0^{a_1} F'(t) g(t)\,dt \geqslant \int_0^{\lambda} F'(t)\,dt,$$

i.e.,

$$\sum_{k=1}^{n} (F(a_{k+1}) - F(a_k))(h_1 - h_2 + \cdots + (-1)^{k-1} h_k) \geqslant F(\lambda) - F(0) \cdot$$

This inequality is equivalent to BRUNK-OLKIN's inequality

$$F(0)\left(1 + \sum_{k=1}^{n} (-1)^{k-1} h_k\right) + \sum_{k=1}^{n} (-1)^{k-1} h_k F(a_k) \geqslant F\left(\sum_{k=1}^{n} (-1)^{k-1} h_k a_k\right).$$

To the best of our knowledge, the following theorem was also proved for the first time by J. F. STEFFENSEN (see: p. 141 of **[18]**):

Theorem 13. *Let g_1 and g_2 be functions defined in $[a, b]$ such that*

$$\int_a^x g_1(t)\,dt \geqslant \int_a^x g_2(t)\,dt$$

for all $x \in [a, b]$ and

$$\int_a^b g_1(t)\,dt = \int_a^b g_2(t)\,dt .$$

Let f be an increasing function on $[a, b]$, then

$$\int_a^b f(x)\, g_1(x)\, dx \leqslant \int_a^b f(x)\, g_2(x)\, dx .$$

If f is a decreasing function on $[a, b]$, then

$$\int_a^b f(x)\, g_1(x)\, dx \geqslant \int_a^b f(x)\, g_2(x)\, dx .$$

Proof. Put $g(x) = g_1(x) - g_2(x)$ and $G(x) = \int_a^x g(t)\, dt$. Then, under the above hypothesis,

$$G(x) \geqslant 0 \quad (a \leqslant x \leqslant b) \quad \text{and} \quad G(a) = G(b) = 0 .$$

Using the STIELTJES integral, we get

$$\begin{aligned}\int_a^b f(t)\, g(t)\, dt &= \int_a^b f(t)\, dG(t) \\ &= f(t)\, G(t)\Big|_a^b - \int_a^b G(t)\, df(t) \\ &= -\int_a^b G(t)\, df(t) .\end{aligned}$$

This proves Theorem 13.

M. MARJANOVIĆ, in **[19]**, considers the above inequality as a special case of a general inequality due to K. FAN and G. G. LORENTZ **[20]** and uses it to give the following short proof.

Let $g_2(x) = g(x)$, $\lambda = \int_a^b g(x)\, dx$ and $g_1(x) = 1$ for $x \in [a, a+\lambda]$ and $g_1(x) = 0$ for $x \in [a+\lambda, b]$.

Then, we have

$$\int_a^{a+\lambda} f(x)\, dx = \int_a^b f(x)\, g_1(x)\, dx \geqslant \int_a^b f(x)\, g(x)\, dx ,$$

which proves the second inequality in (1). One similarly derives the first inequality in (1).

P. VERESS, in **[21]**, uses the technique of the STIELTJES integration to obtain an inequality containing the inequality in Theorem 13, as well as its discrete form.

Now, we shall state some results of Z. CIESIELSKI **[22]**, related to Theorems 2 and 3 of STEFFENSEN. Apparently CIESIELSKI was unaware of STEFFENSEN's results.

Theorem 14. *Let (p_i) denote a sequence of real numbers such that*

$$\sum_{i=1}^{k} p_i \geqslant 0 \text{ for } k=1, \dots, n \text{ and } \sum_{i=1}^{n} |p_i| > 0 .$$

Let $x_i \in [0, a]$ (where a is a positive constant) for $i=1, \dots, n$ and let $x_1 \geqslant \cdots \geqslant x_n$. Further, let $f(x)$ and $f'(x)$ be continuous and convex functions in $[0, a]$ and let $f(0) \leqslant 0$. Then

$$f\left(\frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} |p_i|}\right) \leqslant \frac{\sum_{i=1}^{n} p_i f(x_i)}{\sum_{i=1}^{n} |p_i|} .$$

Theorem 15. *Let the function $g(t)$ be nonincreasing in $[\alpha, \beta]$ and let $a \geqslant g(t) \geqslant 0$ in $[\alpha, \beta]$. Let $f(x)$ and $f'(x)$ be continuous and convex in $[0, a]$ and let $f(0) \leqslant 0$. Further, let $p(t)$ be a function integrable in the Lebesgue sense in $[\alpha, \beta]$, such that*

$$\int_{\alpha}^{x} p(t)\, dt \geqslant 0 \text{ for } x \in [\alpha, \beta] \text{ and } \int_{\alpha}^{\beta} |p(t)|\, dt > 0 .$$

Then

$$f\left(\frac{\int_{\alpha}^{\beta} p(t) g(t)\, dt}{\int_{\alpha}^{\beta} |p(t)|\, dt}\right) \leqslant \frac{\int_{\alpha}^{\beta} p(t) f(g(t))\, dt}{\int_{\alpha}^{\beta} |p(t)|\, dt} .$$

In the same paper analogous results are given for functions of two variables and applications are made to establishing of generalizations of ČEBYŠEV's and BIERNACKI's inequalities **[15]**.

It would be interesting to find interconnections between STEFFENSEN's and CIESIELSKI's generalizations of JENSEN's inequality.

R. APÉRY in his short note **[23]**, which contains no references, proved a varinat of STEFFENSEN's inequality. His result reads:

Theorem 16. *Let $f(x)$ be a monotone decreasing function in $(0, +\infty)$. Let $g(x)$ be a measurable function in $[0, +\infty)$ such that $0 \leqslant g(x) \leqslant A$ (A is a constant $\neq 0$). Then*

$$\int_{0}^{+\infty} f(x) g(x)\, dx \leqslant A \int_{0}^{\lambda} f(x)\, dx ,$$

where

$$\lambda = \frac{1}{A} \int_{0}^{+\infty} g(x)\, dx.$$

Apéry, in his elegant proof, starts from the identity

$$\int_0^{+\infty} f(x)\,g(x)\,dx = A\int_0^{\lambda} f(x)\,dx - \int_0^{\lambda} [A-g(x)]\,[f(x)-f(\lambda)]\,dx$$
$$-\int_{\lambda}^{+\infty} g(x)\,[f(\lambda)-f(x)]\,dx\,.$$

Steffensen's inequalities (1) follow immediately if we use Apéry's idea, namely if we start from the identities

$$\int_a^{a+\lambda} f(t)\,dt - \int_a^{b} f(t)\,g(t)\,dt$$
$$= \int_a^{a+\lambda} [f(t)-f(a+\lambda)]\,[1-g(t)]\,dt + \int_{a+\lambda}^{b} [f(a+\lambda)-f(t)]\,g(t)\,dt$$

and

$$\int_a^{b} f(t)\,g(t)\,dt - \int_{b-\lambda}^{b} f(t)\,dt$$
$$= \int_a^{b-\lambda} [f(t)-f(b-\lambda)]\,g(t)\,dt + \int_{b-\lambda}^{b} [f(b-\lambda)-f(t)]\,[1-g(t)]\,dt.$$

Finally we notice that E. K. Godunova and V. I. Levin **[24]** have recently obtained a general result which contains Steffensen's inequality **[1]**. This, once again, affirms the importance of this inequality which arises from various other inequalities.

In his very interesting and instructive paper **[25]** G. H. Hardy says:

— The really fundamental inequalities are strictly elementary . . . (p. 63);

— An elementary inequality ... carries with it a whole set of analogues and extensions ... (p. 64).

In our opinion the Steffensen inequality is „elementary“ in Hardy's sense, and in articles such as **[24]** it would be more adequate to speak of a generalization of Steffensen's inequality, rather than a general inequality which contains Steffensen's inequalty as a particular case.

Also related to the Steffnsen inequality are references **[26]** and **[27]**.

Added in proof. 1. Though the Steffensen inequality is not included in the source book for inequalities **[14]**, in recent time it is cited even in books dedicated to University studies, as for example, in **[28]**, p. 83, and **[29]**, p. 50. It should be noted that Steffensen's inequality can be found in Bourbaki **[30]**. J. Dieudonné **[29]**, p. 50, gives the Steffensen inequality in the following form:

»Soient f, g deux fonctions continues par morceaux dans $[a, b]$, telles que f soit décroissante et $0 \leqslant g(t) \leqslant 1$ dans $[a, b]$. Si on pose

$$\lambda = \int_a^b g(t)\,dt\,,$$

on a

$$\int_{b-\lambda}^{b} f(t)\,dt \leqslant \int_{a}^{b} f(t)\,g(t)\,dt \leqslant \int_{a}^{a+\lambda} f(t)\,dt,$$

l'égalité ne pouvant avoir lieu que si f est constante dans $[a, b]$ ou si g est égale à 0 sauf en ses points de discontinuité, ou à 1 sauf en ses points de discontinuité«.

2. Roy O. Davies has communicated to us the following interesting proof of the second inequality of (1).

The function H, defined by

$$H(x) = \int_{a}^{a+\int_a^x g(t)\,dt} f(t)\,dt - \int_a^x f(t)\,g(t)\,dt,$$

is zero when $x=0$ and has positive derivative:

$$H'(x) = f\Big(a + \int_a^x g(t)\,dt\Big)\,g(x) - f(x)\,g(x) \geqslant 0$$

since $a + \int_a^x g(t)\,dt \leqslant x$, because of the hypothesis $0 \leqslant g(t) \leqslant 1$, and thus

$$f\Big(a + \int_a^x g(t)\,dt\Big) \geqslant f(x)$$

as f is decreasing.

This holds for smooth functions, and can be extended to others by the usual approximations.

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