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213. ON INEQUALITIES CONNECTING ARITHMETIC MEANS AND GEOMETRIC MEANS OF TWO SETS OF THREE POSITIVE NUMBERS, II.*

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1. In a note [1] with the same title I proved some inequalities about the arithmetic means and geometric means of two sets of three numbers which satisfy the following condition

H: c_1, c_2, c_3 lie between the least and greatest of the three positive numbers a_1, a_2, a_3.

(Without loss of generality we can assume that

$$0 < a_1 \leqslant a_2 \leqslant a_3, \quad 0 < c_1 \leqslant c_2 \leqslant c_3;$$

so that H means $a_1 \leqslant c_1$, $c_3 \leqslant a_3$.)

I. If $c_1 + c_2 + c_3 \geqslant a_1 + a_2 + a_3$, then $c_1 c_2 c_3 \geqslant a_1 a_2 a_3$: equality is possible if and only if $a_i = c_i$ $(i = 1, 2, 3)$.

II. If $a_1 a_2 a_3 \geqslant c_1 c_2 c_3$, then $a_1 + a_2 + a_3 \geqslant c_1 + c_2 + c_3$: equality is possible if and only if $a_i = c_i$ $(i = 1, 2, 3)$.

In this note I show that I can be strengthened but not II.

III. *Suppose that* $0 \leqslant n \leqslant 2$, *that the* c_i, a_i *satisfy* H *and that* $c_1 + c_2 + c_3 \geqslant$ $\geqslant a_1 + a_2 + a_3$. *Then*

$$(a_1 + a_2 + a_3)^n c_1 c_2 c_3 \geqslant (c_1 + c_2 + c_3)^n a_1 a_2 a_3;$$

equality implies equality of the c_i *and* a_i.

For any $n > 2$ *the inequality fails for appropriately chosen* a_i, c_i.

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IV. *Let δ be an arbitrarily small positive number. Numbers c_i, a_i satisfying* H *and $a_1 a_2 a_3 > c_1 c_2 c_3$ exist such that*

$$(a_1 + a_2 + a_3)(c_1 c_2 c_3)^\delta < (c_1 + c_2 + c_3)(a_1 a_2 a_3)^\delta$$

although (by II)

$$a_1 + a_2 + a_3 \geqslant c_1 + c_2 + c_3.$$

2. The negative part of III is settled by the sets

$$(a_i) = (1, G-1, G), \quad (c_i) = (1, G, G) \qquad (G > 2)$$

which satisfy H when we take G sufficiently large.

Let $n = 2 + \varepsilon$, $\varepsilon > 0$. Then

$$\log\{(a_1 + a_2 + a_3)^n c_1 c_2 c_3 / (c_1 + c_2 + c_3)^n a_1 a_2 a_3\} = -\frac{\varepsilon}{G} + O\left(\frac{1}{G^2}\right) < 0$$

if G is sufficiently large.

To prove IV take

$$(a_i) = (1, 9, K), \quad (c_i) = (2, 3, K) \qquad (K \text{ large}).$$

Then

$$\frac{a_1 + a_2 + a_3}{c_1 + c_2 + c_3}\left(\frac{c_1 c_2 c_3}{a_1 a_2 a_3}\right)^\delta = \left(1 + \frac{5}{K+5}\right)\left(\frac{2}{3}\right)^\delta < 1$$

for sufficiently large K.

3. To prove III I assume I and use the inequalities below about two pairs of positive numbers which throughout satisfy the conditions

$$0 < a \leqslant b, \quad 0 < \alpha \leqslant \beta.$$

Lemma 1. If $b\alpha \geqslant a\beta$, then

$$\alpha\beta(a+b)^2 \geqslant ab(\alpha+\beta)^2;$$

equality if and only if $b\alpha = a\beta$.

We have

$$\alpha\beta(a+b)^2 - ab(\alpha+\beta)^2 = (b\beta - a\alpha)(b\alpha - a\beta).$$

But $b\beta - a\alpha \geqslant b\alpha - a\beta \geqslant 0$. The result follows.

Lemma 2. If $b\alpha \geqslant a\beta$ and $\alpha + \beta \geqslant a + b$, then

$$\alpha\beta(a+b)^n \geqslant ab(\alpha+\beta)^n \qquad (0 < n \leqslant 2);$$

equality if and only if $a = \alpha$, $b = \beta$.

For $n = 2$ the result follows from Lemma **1** since $b\alpha \geqslant a\beta$. For $0 \leqslant n < 2$ use $\alpha + \beta \geqslant a + b$. Equality occurs if and only if $b\alpha = a\beta$, $\alpha + \beta = a + b$ whence

$$a = \alpha, \quad b = \beta.$$

Lemma 3. If $0 < a \leqslant \alpha \leqslant \beta \leqslant b$ and $\alpha + \beta \geqslant a + b$, then

$$\alpha\beta(a+b)^n \geqslant ab(\alpha+\beta)^n \qquad (0 \leqslant n \leqslant 2):$$

equality holds if and only if $a = \alpha$, $b = \beta$.

The conditions imply that $b\alpha \geqslant a\beta$ so that Lemma **3** follows from Lemma **2.**

4. *Proof of III.* Two cases arise according as

(i) $$c_1(a_1+a_2+a_3)\geqslant a_1(c_1+c_2+c_3)$$
or
(ii) $$c_1(a_1+a_2+a_3)<a_1(c_1+c_2+c_3).$$

If (i) holds we prove that $c_1+c_2+c_3\geqslant a_1+a_2+a_3$ implies

$$(a_1+a_2+a_3)^3\,c_1c_2c_3\geqslant(c_1+c_2+c_3)^3\,a_1a_2a_3$$

(which is stronger than the inequality in III).

Consider the two sets b_i, d_i defined by

$$b_i=a_i/(a_1+a_2+a_3),\quad d_i=c_i/(c_1+c_2+c_3).$$

Plainly $0<b_1\leqslant b_2\leqslant b_3$, $0<d_1\leqslant d_2\leqslant d_3$, $\Sigma b_i=\Sigma d_i=1$, $d_3\leqslant b_3$ since $d_3\leqslant a_3/\Sigma c\leqslant a_3/\Sigma a=b_3$, $b_1\leqslant d_1$ by the assumption (i).

Thus the b_i, d_i satisfy the conditions of I so that

$$d_1d_2d_3\geqslant b_1b_2b_3,\quad \text{i.e.}\quad (a_1+a_2+a_3)^3\,c_1c_2c_3\geqslant(c_1+c_2+c_3)^3\,a_1a_2a_3$$

as required.

We come now to case (ii). Here necessarily

$$c_1(a_2+a_3)<a_1(c_2+c_3)\leqslant c_1(c_2+c_3)$$

so that

$$a_2+a_3<c_2+c_3\leqslant c_2+a_3,\quad a_2<c_2.$$

Thus the two pairs of positive numbers a_2, a_3; c_2, c_3 satisfy the conditions of Lemma **3** so that

$$c_2c_3(a_2+a_3)^n\geqslant a_2a_3(c_2+c_3)^n\qquad(0\leqslant n\leqslant 2)$$

(with strict inequality since $c_2+c_3>a_2+a_3$). We apply the inequalities to appropriately grouped terms in expansion of

$$(a_1+a_2+a_3)^2\,c_1c_2c_3-(c_1+c_2+c_3)^2\,a_1a_2a_3=L+M+N,$$

where

$$L=c_1a_1(a_1c_2c_3-c_1a_2a_3),$$

$$M=2\,c_1a_1\{c_2c_3(a_2+a_3)-a_2a_3(c_2+c_3)\},$$

$$N=c_1c_2c_3(a_2+a_3)^2-a_1a_2a_3(c_2+c_3)^2.$$

Then (using $a_1(c_2+c_3)>c_1(a_2+a_3)$)

$$L(c_2+c_3)\geqslant c_1a_1c_1\{c_2c_3(a_2+a_3)-a_2a_3(c_2+c_3)\}>0\qquad\text{(Lemma 3)},$$

$$M>0\qquad\text{(Lemma 3)},$$

$$N\geqslant a_1\{c_2c_3(a_2+a_3)^2-a_2a_3(c_2+c_3)^2\}>0\qquad\text{(Lemma 3)}.$$

Thus case (ii) is settled: if $c_1(a_1+a_2+a_3)<a_1(c_1+c_2+c_3)$, then III holds with strict inequality.

5. *An application of III.*

Suppose that ABC, $A'B'C'$ are two triangles such that (i) the perimeter of $A'B'C'$ is at least equal to the perimeter of ABC, (ii) the lengths of the tangents from A', B', C' to the incircle of $A'B'C'$ lie between the least and greatest of the tangents from A, B, C to the incircle of ABC.

Then

(1) $$\text{inradius of } A'B'C' \geqq \text{inradius of } ABC,$$

(2) $$\text{area of } A'B'C' \geqq \text{area of } ABC.$$

Equality occurs if and only if ABC, $A'B'C'$ are congruent.

Taking $A'B'C'$ to be an equilateral triangle of the same perimeter as ABC yields the well-known inequality

$$S^2 \geqslant 3\sqrt{3}\,\Delta,$$

where S is the semi-perimeter and Δ the area of ABC.

Equality holds if and only if ABC is equilateral.

REFERENCE

[1] A. Oppenheim, *On inequalities connecting arithmetic means and geometric means of two sets of three positive numbers*, Math. Gazette, **49** (1965), 160—162.

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