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THE MONOTONICITY OF $I_1(x) / I_0(x)$*

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In fitting a »circular normal distribution«

$$f(t \mid x, \theta) = \frac{\exp\{x \cos(t-\theta)\}}{2\pi I_0(x)} \quad \text{where } 0 \leqslant t < 2\pi, \ 0 \leqslant \theta < 2\pi$$

and $x > 0$ to a set of readings t_r, ($r = 1$ to n), the maximum likelihood estimate of the parameter x is given (see [1]) by the equation

$$\frac{I_1(x)}{I_0(x)} = \frac{1}{n} \sqrt{s^2 + c^2}$$

where $s = \sum_{r=1}^{n} \sin t_r$ and $c = \sum_{r=1}^{n} \cos t_r$.

Anumerical table of $g(x) = I_1(x) / I_0(x)$ prepared by GUMBEL, GREENWOOD and DURAND [1] suggests that this function is increasing for positive x. If, for such x, it is monotonic with range 0 to 1 then the existence of a unique maximum likelihood estimate for x is ensured. The following is a direct verification of this behaviour of the function $g(x)$.

Putting $t = e^{i\theta}$ in the LAURENT series

$$\sum_{n=-\infty}^{+\infty} I_n(x) t^n = e^{(t+1/t) x/2}$$

and using

$$I_{-n}(x) = I_n(x)$$

gives

$$I_0(x) + 2 \sum_{n=1}^{\infty} I_n(x) \cos n\theta = e^{x \cos \theta}.$$

By regarding the left hand side as a half-range cosine expansion of $e^{x \cos \theta}$ for the range $(-\pi, \pi)$ we get that

$$I_n(x) = \frac{1}{\pi} \int_0^{\pi} e^{x \cos \theta} \cos n\theta \, d\theta.$$

* Presented December 5, 1967 by D. S. Mitrinović.

This can also be obtained by expanding the integrand in powers of x and integrating term by term. For either method the justification is straightforward, as is that for differentiation under the integral sign to yield

$$I_n'(x)=\frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta}\cos\theta\cos n\theta\, d\theta.$$

Then

$$\pi^2\{I_0(x)\,I_1'(x)-I_1^2(x)\}=\int_0^{\pi} e^{x\cos\theta}\,d\theta\times\int_0^{\pi} e^{x\cos\theta}\cos^2\theta\,d\theta-\Big\{\int_0^{\pi} e^{x\cos\theta}\cos\theta\,d\theta\Big\}^2$$

$$\geqslant\Big\{\int_0^{\pi} e^{x\cos\theta}\,|\cos\theta|\,d\theta\Big\}^2-\Big\{\int_0^{\pi} e^{x\cos\theta}\cos\theta\,d\theta\Big\}^2,$$

by HÖLDER'S inequality

$$=\int_0^{\pi} e^{x\cos\theta}(|\cos\theta|-\cos\theta)\,d\theta\times\int_0^{\pi} e^{x\cos\theta}(|\cos\theta|+\cos\theta)\,d\theta$$

>0 for all x.

Hence, since $I_0'(x)=I_1(x)$ and $I_0(x)>0$, we have

$$g'(x)=\frac{I_0(x)\,I_1'(x)-I_1^2(x)}{I_0^2(x)}>0\,.$$

Since $I_0(0)=1$, $I_1(0)=0$ and $I_0(x)\sim e^x/\sqrt{2\pi x}$, $I_1(x)\sim e^x/\sqrt{2\pi x}$ as $x\to\infty$, it now follows that $g(x)$ is monotonic increasing for $x>0$ and that in this semi-infinite range it assumes each value between 0 and 1 exactly once.

In view of the inequality $I_n(x)>I_{n+1}(x)$ for $n>-\frac{1}{2}$, $x>0$, proved by SONI [2], it would be of interest to know if the above result can be generalized to proving the monotonicity of $I_{n+1}(x)/I_n(x)$.

REFERENCES

[1] E. J. GUMBEL, J. A. GREENWOOD and D. DURAND, *The circular normal distribution; theory and tables*, Jour. Amer. Stat. Assn. 48 (1953), 131—152.

[2] R. P. SONI, *On an inequality for modified Bessel functions*, Jour. of Math. and Physics 44 (1965), 406—407.