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AN IMPROPER INTEGRAL*

Radomir P. Lučić

Let us consider the integrals

$$I=\int_{-\infty}^{+\infty}\frac{1}{((x-a_1)^2+b_1^2)\cdots((x-a_n)^2+b_n^2)}\,dx \tag{1}$$

where $b_k>0$, a_k real, $k=1, \ldots, n$; $(a_k, b_k)\neq(a_m, b_m)$ for $k\neq m$, and

$$J=\oint_C f(z)\,dz$$

where $C=K\cup[-R, R]$ and $K=\{z:|z|=R\wedge I_m(z)>0,\ R>\max\sqrt{a_k^2+b_k^2}\}$,

$$f(z)=\frac{1}{\prod\limits_{k=1}^{n}((z-a_k)^2+b_k^2)}.$$

Being

$$I=\lim_{R\to+\infty}\int_{-R}^{+R}f(x)\,dx,\qquad J=\int_K f(z)\,dz+\int_{-R}^{+R}f(x)\,dx,$$

$$\lim_{R\to+\infty}\int_K f(z)\,dz=0\quad\text{and}\quad J=2\pi i\sum_{k=1}^{n}\operatorname{res}f(z_k),\ z_k=a_k+ib_k$$

we have

$$I=2\pi i\sum_{k=1}^{n}\operatorname{res}f(z_k). \tag{2}$$

Since

$$\operatorname{res}f(z_k)=\frac{1}{2b_k i\prod\limits_{\substack{j=1\\j\neq k}}^{n}((a_k+ib_k-a_j)^2+b_j^2)}=\frac{1}{2b_k i\prod\limits_{\substack{j=1\\j\neq k}}^{n}(m_{j,k}+in_{j,k})},$$

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where $m_{j,k}=(a_j-a_k)^2+b_j^2-b_k^2$, $n_{j,k}=2b_k(a_k-a_j)$ and since the integral (1) is real, it follows from (2) that

$$I=\pi\sum_{k=1}^{n}\frac{1}{b_k}\operatorname{Re}\frac{1}{\prod\limits_{\substack{j=1\\ j\neq k}}^{n}(m_{j,k}+in_{j,k})}=\pi\sum_{k=1}^{n}\frac{\operatorname{Re}\left(\prod\limits_{\substack{j=1\\ j\neq k}}^{n}(m_{j,k}+in_{j,k})\right)}{b_k\prod\limits_{\substack{j=1\\ j\neq k}}^{n}(m_{j,k}^2+n_{j,k}^2)}.$$

So, we have further

$$I=\pi\sum_{k=1}^{n}\frac{1}{b_k\prod\limits_{\substack{j=1\\ j\neq k}}^{n}(m_{j,k}^2+n_{j,k}^2)}\sum_{p=0}^{\left[\frac{n-1}{2}\right]}(-1)^p\sum n_{i_1,k}\ldots n_{i_{2p},k}\,m_{j_1,k}\ldots m_{j_{n-2p},k}$$

where

$$\sum n_{i_1,k}\ldots n_{i_{2p},k}\,m_{j_1,k}\ldots m_{j_{n-2p},k}$$

stands for the sum of all combinations of the form

$$n_{i_1,k}\ldots n_{i_{2p},k}\,m_{j_1,k}\ldots m_{j_{n-2p},k}$$

in which $i_k\neq j_s$ and $1\leqslant i_k,\ j_s\leqslant n$ excluding the cases $i_r=k$ and $j_t=k$ $(r=1,2,\ldots,2p;\ t=1,2,\ldots,n-2p)$.

In [1], the integral (1) has been considered in the cases when $n=1,2,3$. For $n=4$, it follows that

$$\begin{aligned}I/\pi=&\frac{m_{2,1}m_{3,1}m_{4,1}-m_{2,1}n_{3,1}n_{4,1}-m_{3,1}n_{2,1}n_{4,1}-m_{4,1}n_{2,1}n_{3,1}}{b_1\prod\limits_{j=2}^{4}(m_{j,1}^2+n_{j,1}^2)}\\&+\frac{m_{1,2}m_{3,2}m_{4,2}-m_{1,2}n_{3,2}n_{4,2}-m_{3,2}n_{1,2}n_{4,2}-m_{4,2}n_{1,2}n_{3,2}}{b_2\prod\limits_{\substack{j=1\\ j\neq 2}}^{4}(m_{j,2}^2+n_{j,2}^2)}\\&+\frac{m_{1,3}m_{2,3}m_{4,3}-m_{1,3}n_{2,3}n_{4,3}-m_{2,3}n_{1,3}n_{4,3}-m_{4,3}n_{1,3}n_{2,3}}{b_3\prod\limits_{\substack{j=1\\ j\neq 3}}^{4}(m_{j,3}^2+n_{j,3}^2)}\\&+\frac{m_{1,4}m_{2,4}m_{3,4}-m_{1,4}n_{2,4}n_{3,4}-m_{2,4}n_{1,4}n_{3,4}-m_{3,4}n_{1,4}n_{2,4}}{b_4\prod\limits_{j=1}^{3}(m_{j,4}^2+n_{j,4}^2)}.\end{aligned}$$

In [2], the integral (1) has been considered in the cases when $a_k=0$, $k=1,2,\ldots,n$.

REFERENCES

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[2] R. P. Lučić: *On an Integral*, Math. Gazette 50 (1966), 392 — 393.