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SOME INEQUALITIES FOR A SPHERICAL TRIANGLE AND AN INTERNAL POINT*

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1. There are many inequalities concerned with a triangle and an internal point, e.g. (i) inequality of ERDÖS-MORDELL connecting the distances (x, y, z) of an internal point O from the vertices A, B, C with the perpendicular distances (p, q, r) from the sides of ABC:

$$x+y+z \geqslant 2(p+q+r) \tag{1}$$

with equality in the single case when O is the centre of an equilateral triangle ABC; (ii) BARROW's inequalitiy

$$x+y+z \geqslant 2(u+v+w) \tag{2}$$

where $u, v\, w$ are the distances of O from BC, CA, AB measured along the respective internal bisectors of the angles BOC, COA, AOB; equality occurs as in (1).

Such inequalities have immediate analogues for a spherical triangle ABC and an internal point O provided that each of the arc distances OA, OB, OC is less than a quadrant. It is enough to replace x by $\tan X$, p by $\tan P$, u by $\tan U$ where X denotes the arc OA, P denotes the perpendicular arc from O to BC, U denotes the arc distance of O to BC measured along the internal bisector of the angle BOC.

The proof is obvious. Project from the centre of the (unit) sphere on the tangent plane at O. We obtain plane triangle $A'B'C'$ with O an interrnal point. Plainly $OA' = \tan OA, \ldots$ And if L is the foot of the perpendicular from O on arc BC, then L' is the foot of the perpendicular from O on side $B'C'$. A corresponding statement is true for the feet of the internal bisectors. Hence in full the following

Theorem 1. *Suppose that O is an internal point of the spherical triangle ABC and that each of the arcs OA, OB, OC is less than a quadrant.*

Then

$$\tan X+\tan Y+\tan Z \geqslant 2(\tan P+\tan Q+\tan R) \tag{3}$$

$$\tan X+\tan Y+\tan Z \geqslant 2(\tan U+\tan V+\tan W)\,, \tag{4}$$

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where P, Q, R are the perpendicular distances of O from the sides BC, CA, AB and U, V, W are the distances of O from these sides measured along the respective internal angle-bisectors of BOC, COA, AOB. Equality holds if and only if O is the centre of an equilateral triangle ABC.

In [1, 2] I showed that BARROW's inequality (2) can be extended to

$$\lambda x + \mu y + \nu z \geqslant 2 \sum \frac{\dfrac{1}{y} + \dfrac{1}{z}}{\dfrac{1}{\mu y} + \dfrac{1}{\nu z}}\, u \tag{5}$$

valid for all positive λ, μ, ν; equality holds if and only if each of the angles BOC, COA, AOB is $\frac{2\pi}{3}$ and $\lambda x = \mu y = \nu z$. Hence

Theoreme 2. *If O is an internal point of the spherical triangle ABC and each of the arcs $X = OA$, $Y = OB$, $Z = OC$ is less than a quadrant then for all positive λ, μ, ν (such that $\lambda X, \mu Y, \nu Z$ are less than a quadrant)*

$$\sum \tan \lambda X \geqslant 2 \sum \frac{\cot Y + \cot Z}{\cot \mu Y + \cot \nu Z} \tan U; \tag{6}$$

equality occurs if and only if each of the angles at O is $\frac{2\pi}{3}$ and

$$\lambda X = \mu Y = \nu Z .$$

Here are two deductions from (6):

$$\sum \tan \frac{X}{2} \geqslant \sum \left(1 - \tan \frac{Y}{2} \tan \frac{Z}{2}\right) \tan U \tag{7}$$

$$\sum \lambda X \geqslant 2 \sum \frac{\cot Y + \cot Z}{\dfrac{1}{\mu Y} + \dfrac{1}{\nu Z}} \tan U \tag{8}$$

the first by $\lambda = \mu = \nu = \frac{1}{2}$; the second by substituting $\lambda\delta, \mu\delta, \nu\delta$ for λ, μ, ν dividing by δ and letting $\delta \to +0$.

Naturally in (6), (7), (8) U, V, W may be replaced by P, Q, R. In addition (7), (8) give rise to curious plane inequalities for a triangle and an internal point.

It is worth noting also that the ternary quadratic form

$$\sum \xi^2 \tan X - 2 \sum \eta\xi \tan U \tag{9}$$

is non-negative definite (O internal point of spherical triangle ABC, arcs X, Y, Z less than quadrants). Hence of course (4) and other inequalities. The truth of (9) follows from the corresponding statement in the plane: the ternary quadratic form

(10) $$X\xi^2+Y\eta^2+Z\zeta^2-2\,U\eta\zeta-2\,V\zeta\xi-2\,W\xi\eta$$

is non-negative definite (as was independently observed by MORDELL [3]).

For the plane figure we have

(11) $$\left(\frac{1}{y}+\frac{1}{z}\right)u=2\cos\alpha \qquad (2\,\alpha=BOC)$$

so thant in the spherical figure

(12) $$(\cot Y+\cot Z)\tan U=2\cos\alpha\,.$$

Since $\sum\xi^2-2\sum\eta\zeta\cos\alpha$ (α, β, γ acute angles sum π) is non-negative definite (and zero if and only if $\xi:\eta:\zeta=\sin\alpha:\sin\beta:\sin\gamma$) both (10) and (9) follow at once.

2. CARLITZ [4] has given a number of inequalities for a triangle ABC and an internal point O, connecting OA, OB, OC with OL, OM, ON where L, M, N are the points in which AO, BO, CO cut BC, CA, AB respectively. Plainly analogues of these inequalities can be written down at once for the spherical triangle by the rule already given.

3. The inequalities found so far for a spherical triangle ABC and an internal point O are restricted by the condition that the arcs OA, OB, OC be less than a quadrant. I give now some inequalities not so resticted but to do this I must allow the sides of ABC to enter.

Theorem 3. *Suppose that O is an internal point of the spherical triangle ABC. Then*

(13) $$\sin X+\sin Y+\sin Z\geqslant 2\sin U\cos\frac{a}{2}+2\sin V\cos\frac{b}{2}+2\sin W\cos\frac{c}{2}\,,$$

(14) $$\sin X+\sin Y+\sin Z\geqslant 2\sin P\cos\frac{a}{2}+2\sin Q\cos\frac{b}{2}+2\sin R\cos\frac{c}{2}\,.$$

Equality occurs in (13) *in any one of the following cases*

(i) *ABC equilateral with centre O.*

(ii) $\pi-X=Y=Z=\delta$, $0<\delta<\pi$ $\left(\textit{and then } V=W=\frac{\pi}{2}\textit{; the other ends of } U, V, W \textit{ are the midpoints of } BC, CA, AB \textit{ respectively}\right)$.

(iii) $X=\pi-Y=Z=\delta$, $0<\delta<\pi$.

(iv) $X=Y=\pi-Z=\delta$, $0<\delta<\pi$.

Apart from these cases there is strict inequality.

To prove (13) note that (12) leads to the equality

$$(\sin^2 Y+\sin^2 Z+2\sin Y\sin Z\cos\alpha)^{\frac{1}{2}}\sin U=2\sin Y\sin Z\cos\alpha \tag{15}$$

(I omit the details). But

$$\sin^2 Y+\sin^2 Z+2\sin Y\sin Z\cos a\geqslant(\sin Y+\sin Z)^2\cos^2\frac{a}{2}$$
$$\geqslant 4\sin Y\sin Z\cos^2\frac{a}{2},$$

so that

$$2\sum\sin U\cos\frac{a}{2}\leqslant 2\sum(\sin Y\sin Z)^{\frac{1}{2}}\cos\alpha\leqslant\sum\sin X$$

as required. Equality occurs if and only if $\alpha=\beta=\gamma=\frac{2}{3}\pi$ and

$$\sin X=\sin Y=\sin Z$$

whence the statement in Theorem 3.

It will be noted that we have also proved the non-negative definitess of the form

$$\sum\xi^2\sin X-2\sum\eta\zeta\sin U\cos\frac{a}{2}. \tag{16}$$

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