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A CLASS OF POLYNOMIAL FILTERS HAVING CONTROLLABLE GROUP-DELAY AND MAGNITUDE CHARACTERISTICS

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SUMMARY: A new class of low-pass filters with all transmission zeros at infinity is described in this paper. The transfer function of the filters is derived algebraicly from Bessel polynomials by using a well-known recurrence formula for orthogonal polynomials and introducing a variabile parameter which controls the even derivatives of the magnitude and group delay responses of the network at the origin. The transient and steady-state responses of this class of filters for various values of the controllable parameter are studied and compared with other known systems. It is shown that these filters, referred to as quasi-Bessel filters, provide an improvement over the Butterworth, Transitional Butterworth-Thomson and other types of filters including that due to Ghausi and Adamowicz, especially in those applications where the rise-time, overshoot and bandwidth of the magnitude response are all of importance.

Introduction

A great deal has been written on the characteristics of the low-pass filters with all transmission zeros at infinity, which are designed to have either maximally flat magnitude or maximally flat group delay responses. In pulse applications, Butterworth filters [1], approximating the uniform magnitude characteristic in the maximally flat manner, have fast rise-time in response to a unit step input, but exibit excessive transient overshoot which increases with the increase in the order of the filter. On the other hand, Thomson filters [2], which have maximally flat group delay response, are characterized by small transient overshoot, but the rise-time is exceedingly slow when compared to that for the Butterworth filters.

Subsequently, other classes of low-pass filters have been studied by various authors in order to obtain a more favorable transient characteristic than that of either the Butterworth or Thomson networks. Mullick [3] has investigated a class of ladder filters with parabolic distribution of poles in the complex frequency plane, while Scanlan [4] proposed a similar class of transfer functions with all zeros at infinity and poles located on an ellipse with equal spacings along the imaginary axis. Both these filters give an improvement over Thomson networks since they provide greater bandwidth while retaining a fairly good approximation to the ideel group delay characteristic in the passband. In the so-called transitional Butterworth-Thomson filters, which have been described by Peless and Murakami [5], the pole locations of the transfer function in the complex frequency plane can be varied smoothly from those of the Butterworth to those of the Thomson filters by varying a parametar which controls the pole positions. Although a large variety of filter specifications can be satisfied by an appropriate choice of the variabile parametar, this class of filters offers no advantages over Butterworth or Thomson filters in respect of transient responses. Quite recently, Ghausi and Adamowicz [6] have described low-pass filters, the transfer functions of which have their poles located on catenary curves in the complex frequency plane at equal angular spacings, and found that this class of filters offers some advantages when the rise-time, overshoot and 3 db frequency of the magnitude response (with no frepuency peaking) are all of importance simultaneously.

In this paper a new class of polynomial filters is derived by a purely analytical approach to the problem of determining the suitable transfer function. The transfer functions of these filters are generated from a class of polynomials which have been obtained from Bessel polynomials by using a reccurence formula and introducing a variabile parametar which controls the values of the even-ordered derivatives of the magnitude and group delay characteristics with respect to ω at the origin. Of course, the odd-ordered derivatives are automatically zero at $\omega=0$ because the magnitude and group delay are even functions of ω. These polynomials have previously been derived and used by the present authors in a study of the problem of phase equalization [7].

The general expressions for the normalized transfer function, magnitude and group delay characteristics are derived and the final results are presented in the form of tables and curves which give the data on the bandwidth, rise-time, overshoot and pole locations for filters of the order $n=3-6$, and different values of the variabile parameter. The comparison shows that the results obtained with these filters are competitive with those of other known systems, especially when fast rise-time with only moderate overshoot and greater bandwidth are required. For example, these filters yield a transient response with equal rise-time and smaller overshoot than the Butterworth-Thomson filters. When compared on equal overshoot basis, they provide shorter rise-time than the Butterworth filters. They also show an improvement over the class of filters proposed by Ghausi and Adamowicz both in the transient response and in 3 db frequency of the magnitude response.

Transfer Functions of the Filter

Bassel polynomials, first investigated by Krall and Fink [8], and then used by Storch [9] in describing the characteristics of the ladder networks with maximally flat group delay response, are defined by

$$y_n(x)=\sum_{k=0}^{n}\frac{(n+k)!}{(n-k)!\,k!}\left(\frac{x}{2}\right)^k \tag{1}$$

from which the polynomials used by Storch are obtained by substituting $x=\frac{1}{s}$ and then defining a new set of polynomials $B_n(s)=s^n y_n\left(\frac{1}{s}\right)$

$$B_n(s)=\sum_{k=0}^{n}\frac{(2n-k)!\,s^k}{2^{n-k}\,k!\,(n-k)!}=\sum_{k=0}^{n}b_k\,s^k \tag{2}$$

Using the well-known recurrence relation for the orthogonal polynomials

$$B_n^{\cdot}(s) = (2n-1)\, B_{n-1} + s^2 B_{n-2} \tag{3}$$

and substituting a new variable p for $(2n-1)$ in Eq. 3, we obtain a new polynomial function

$$B_n(s,p) = p\, B_{n-1}(s) + s^2 B_{n-2}(s) \tag{4}$$

The transfer function of the n-th order ladder filter can now be defined as

$$F_n(s) = \frac{b_0 p}{p\, B_{n-1}(s) + s^2 B_{n-2}(s)} \tag{5}$$

where $b_0 = \dfrac{(2n-2)!}{2^{n-1}(n-1)!}$.

For $B_n(s,p)$ to be a Hurwitz polynomial the necessary and sufficient condition is $p > 0$.

General expressions for Magnitude and Group-Delay Characteristics

The transfer function, Eq. 5, can be transfered in a more suitable form for determining the general expressions of the magnitude and group-delay characteristics by means of the following formula connecting the Bessel polynomials and Bessel functions of half-integral order [8]

$$y_n\left(\frac{1}{j\omega}\right) = j^{-n} e^{j\omega} \left[(-1)^n J_{-(n-1/2)}(\omega) - J_{n-1/2}(\omega)\right] \tag{6}$$

Thus we obtain from Eqs. 2, 5 and 6 and substituting $s = j\omega$

$$F_n(j\omega) = \tag{7}$$

$$= \frac{b_0 p\, e^{j\omega}}{\omega^{n-1}\sqrt{\frac{\pi\omega}{2}}\left\{(-1)^n \left[p J_{-(n-1/2)}(\omega) + \omega J_{-(n-3/2)}(\omega)\right] - j\left[p J_{n-1/2}(\omega) - \omega J_{n-3/2}(\omega)\right]\right\}}$$

from which the magnitude and group delay characteristics can be easely derived

$$A_n(\omega) =$$

$$\frac{b_0 p}{\sqrt{\frac{\pi}{2}\omega^{2n-1}}\left\{\left[p J_{-(n-1/2)}(\omega) + \omega J_{-(n-3/2)}(\omega)\right]^2 + \left[p J_{n-1/2}(\omega) - \omega J_{n-3/2}(\omega)\right]^2\right\}^{1/2}} \tag{8}$$

$$D_n(\omega) = 1 - \frac{d}{d\omega}\left[\mathrm{tg}^{-1}(-1)^{n-1} \frac{p J_{n-1/2}(\omega) - \omega J_{n-3/2}(\omega)}{p J_{-(n-1/2)}(\omega) + \omega J_{-(n-3/2)}(\omega)}\right] \tag{9}$$

Further useful transformations of the last two expressions can be obtained by using some relationships between Bessel functions of half-integral order and the Lommel polynomials which are defined by (10)

$$R_{2m,\,1/2-m}(\omega)=\sum_{0}^{m}\frac{(-1)^k(2m-k)!\,\Gamma\left(\frac{1}{2}+m-k\right)\left(\frac{1}{2}\omega\right)^{2m+2k}}{k!\,(2m-2k)!\,\Gamma\left(\frac{1}{2}-m+k\right)} \tag{10}$$

Using the recurrence formulas for Lommel polynomials

$$J_{\nu+m}(\omega)=J_\nu(\omega)\,R_{m,\nu}(\omega)-J_{\nu-1}(\omega)\,R_{m-1,\,\nu-1}(\omega) \tag{11}$$

$$(-1)^m J_{-\nu-m}(\omega)=J_{-\nu}(\omega)\,R_{m,\nu}(\omega)+J_{-\nu+1}(\omega)\,R_{m-1,\,\nu+1}(\omega) \tag{12}$$

and the following relationship

$$J^2_{m+1/2}(\omega)+J^2_{-(m+1/2)}(\omega)=2(-1)^m\,\frac{R_{2m,\,1/2-m}(\omega)}{\pi\omega} \tag{13}$$

we obtain, after some manipulation, the expression for the magnitude squared function in the following form

$$A_n^2(\omega)=\frac{c_0}{\sum\limits_{k=0}^{n}c_k\,\omega^{2k}} \tag{14}$$

where

$$c_k=(-1)^{n-1-k}(2n-2-k)!\left[\binom{-1/2}{n-1-k}\frac{p^2}{\Gamma(k+1)}-\right.$$
$$\left.-\binom{-1/2}{n-1-k}\frac{2p}{\Gamma(k)}-\binom{-1/2}{n-k}\frac{1}{\Gamma(k-1)}\right] \tag{15}$$

Similarly, the following recurrence formulas for the Bessel functions

$$\omega J_\nu'(\omega)+\nu J_\nu(\omega)=\omega J_{\nu-1}(\omega) \tag{16}$$

$$\omega J_\nu'(\omega)-\nu J_\nu(\omega)=-\omega J_{\nu+1}(\omega) \tag{17}$$

and the Lommel formulas (10)

$$J'_{-\nu}(\omega)\,J_\nu(\omega)-J_\nu'(\omega)\,J_{-\nu}(\omega)=-\frac{2\sin\nu\pi}{\pi\omega} \tag{18}$$

$$J_\nu(\omega)\,J_{1-\nu}(\omega)+J_{-\nu}(\omega)\,J_{\nu-1}(\omega)=\frac{2\sin\nu\pi}{\pi\omega} \tag{19}$$

are employed to obtain the group delay characteristic in the form

$$D_n(\omega) = 1 - \frac{\omega^{2n-2}\,[p^2-(2n-1)\,p+\omega^2]}{\sum_{k=0}^{n} c_k\,\omega^{2k}} \tag{20}$$

The magnitude squared function Eq. 14 is analitic at the origin and may be expanded into a Maclaurin series. Obtaining this expansion by long division and retaining only the first few terms we get

$$A_n^2(\omega) = \frac{c_0}{c_0 + c_1\omega^2 + c_2\omega^4 + \cdots + \omega^{2n}} = 1 - \frac{c_1}{c_0}\,\omega^2 + \left(\frac{c_1^2}{c_0^2} - \frac{c_2}{c_0}\right)\omega^4 + \cdots =$$
$$= 1 - \frac{p-2}{p\,(2n-3)}\,\omega^2 + \frac{n-3}{(2n-3)^2\,(2n-5)\,p^2}\left(p^2 - 4p + \frac{3\,(2n-5)}{n-3}\right)\omega^4 + \cdots \tag{21}$$

In a similar way, the first few terms of the series expansion for the group delay function, Eq. 20, are obtained

$$D_n(\omega) = 1 - \frac{p-(2n-1)}{p\,[(2n-3)!!]^2}\,\omega^{2n-2} + \frac{p^2-(2n-1)\,p+2n+1}{2n-3}\,\omega^{2n} + \cdots \tag{22}$$

where

$$(2n-3)!! = 1\cdot 3\cdot 5\cdots(2n-5)\,(2n-3)$$

The lowest power of ω in the Maclaurin series of the group delay function is ω^{2n-2}, which means that the first $2n-3$ derivatives are equal to zero at the origin. If $p=2n-1$, the coefficient of ω^{2n-2} is also equal to zero and the maximally flat delay characteristic is obtained. On the other hand, all the powers of ω^2 are present in the Maclaurin series of the squared magnitude function (Eq. 21). For maximally flat delay approximation $(p=2n-1)$, the coefficient of ω^2 in the series expansion of the squared magnitude is equal $1/p=1/(2n-1)$. On the other side, if $p=2$, the coefficient of ω^2 in Eq. 21 is zero, i. e., one condition for the flat magnitude response is obtained in addition to the $(n-2)$ similar flatness conditions imposed on the group delay function. The class of polynomial filters satisfying these conditions was studied by Golay [11]. He found that these filters have magnitude and delay characteristics which are quite flat for lower frequencies but have undesirable peaks at the end of the passband. Since these peaks are excessive, especially for higher order filters, they cause a large transient overshoot, so that the practical values of p are higher than 2.

Steady-State and Transient Characteristics of the Filters

The derived equations have then been used to calculate the magnitude and group delay responses of the filters for order n = 3—6 and for different values of the parameter p. The unit-step responses of the filters have also been calculated by standard method and the values of rise-time (10 to 90 percent) and overshoot are summarized. In order to enable the comparison of these filters with similar systems, the DC gain is taken to be identical and equal to unity in all cases.

Third-order Filters

The transfer function of the third order filter, as obtained from Eq. 5, is

$$F_3(s) = \frac{3p}{3p + 3ps + (p+1)s^2 + s^3} \tag{23}$$

From Eqs. 14, 15 and 20 the magnitude and group delay characteristics are obtained in the following forms

$$A_3(\omega) = \frac{3p}{\left[\sum_0^3 c_k \omega^{2k}\right]^{1/2}} \tag{24}$$

$$D_3(\omega) = 1 - \frac{\omega^4 (p^2 - 5p + \omega^2)}{\sum_0^3 c_k \omega^{2k}} \tag{25}$$

where

$$\sum_0^3 c_k \omega^{2k} = 9p^2 + (3p^2 - 6p)\omega^2 + (p^2 - 4p + 1)\omega^4 + \omega^6$$

The steady-state and transient responses of the third order filters for several different values of the parameter p are show in Figs. 1, 2 and 3, while Table I contains the data on the normalized pole locations, the normalized 3 db bandwidth (B), the normalized rise-time (τ) and the percent overshoot (γ).

TABLE I

THIRD-ORDER FILTERS

p	2.5	3.0	3.2	3.5	4.0	4.0
s_1	−0.8627	−0.8573	−0.8576	−0.8607	−0.8736	−0.8985
s_2, s_3	−0.4627	−0.5329	−0.5593	−0.5971	−0.6552	−0.7057
	$\pm j$ 0.9722	$\pm j$ 0.9394	$\pm j$ 0.9237	$\pm j$ 0.8973	$\pm j$ 0.8458	$\pm j$ 0.7842
B	1.10	1.00	0.98	0.93	0.85	0.78
τ	2.15	2.33	2.41	2.53	2.70	2.89
γ%	8.37	5.21	4.27	3.15	1.87	1.13

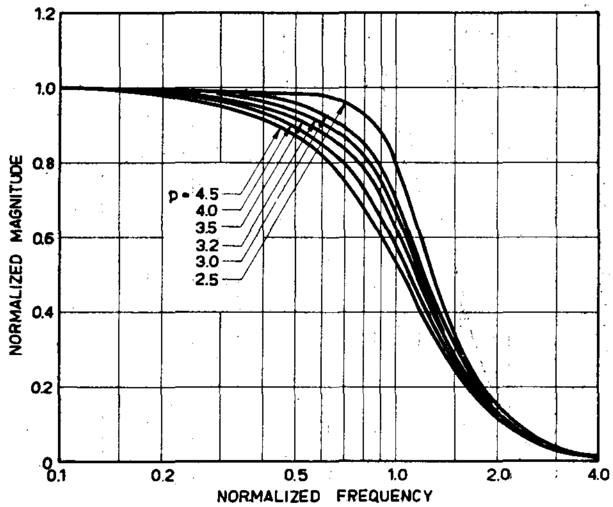

Fig. 1 — Magnitude responses of the 3rd order filters

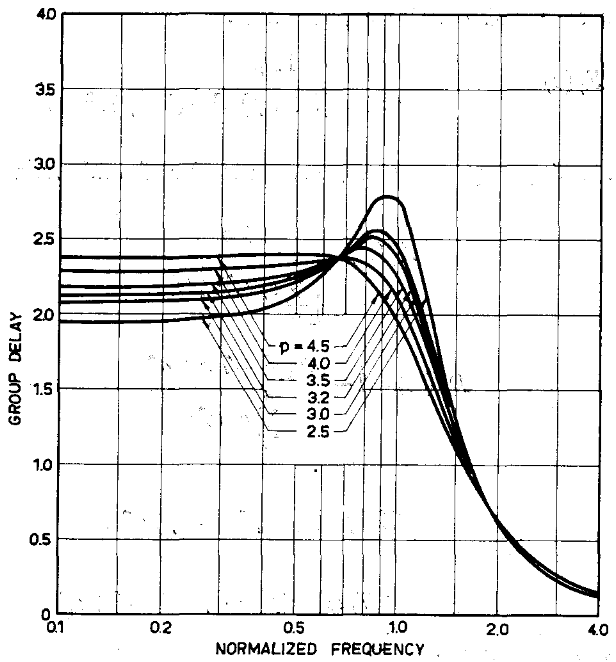

Fig. 2 — Group delays of the 3rd order filters

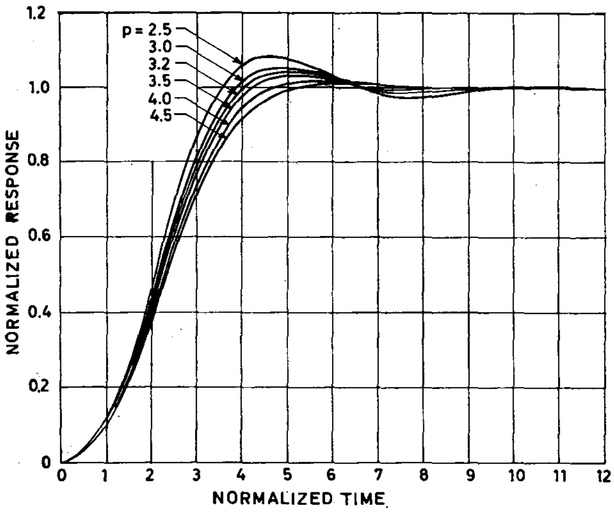

Fig. 3 — Step responses of the 3rd order filter

Fourth-Order Filters

The fourth order transfer function may be found in the same way as in the previous case and can be expressed as

$$F_4(s) = \frac{15p}{15p + 15ps + 3(1+2p)s^2 + (3+p)s^2 + s^4} \tag{26}$$

The magnitude and group delay responses can be found from Eqs. 14, 15 and 20

$$A_4(\omega) = \frac{15p}{\left[\sum_0^4 c_k \omega^{2k}\right]^{1/2}} \tag{27}$$

$$D_4(\omega) = 1 - \frac{\omega^6 (p^2 - 7p + \omega^2)}{\sum_0^4 c_k \omega^{2k}} \tag{28}$$

where

$$\sum_0^4 c_k \omega^{2k} = 225p^2 + (45p^2 - 90p)\omega^2 +$$

$$+ (6p^2 - 24p + 9)\omega^4 + (p^2 - 6p + 3)\omega^6 + \omega^8$$

The magnitude, group delay and trasient responses for various values of p are presented in Figs. 4—6. The pole positions and other important parameters, the normalized rise-time (τ), the per cent overshoot (γ), and the normalized 3 db bandwidth (B) are given in Table II.

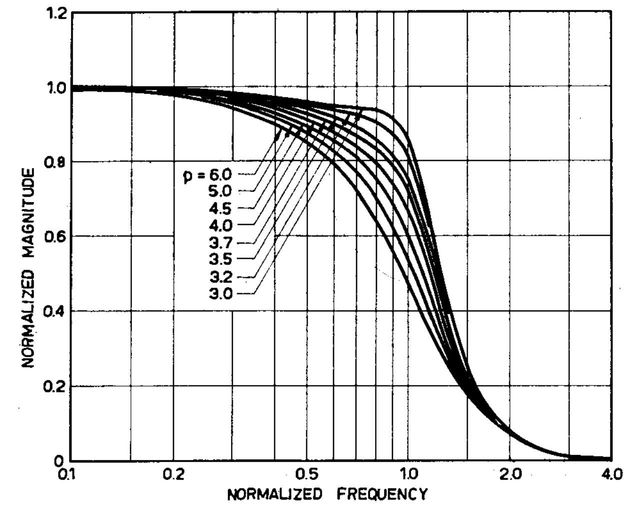

Fig. 4 — Magnitude responses of the 4th order filters

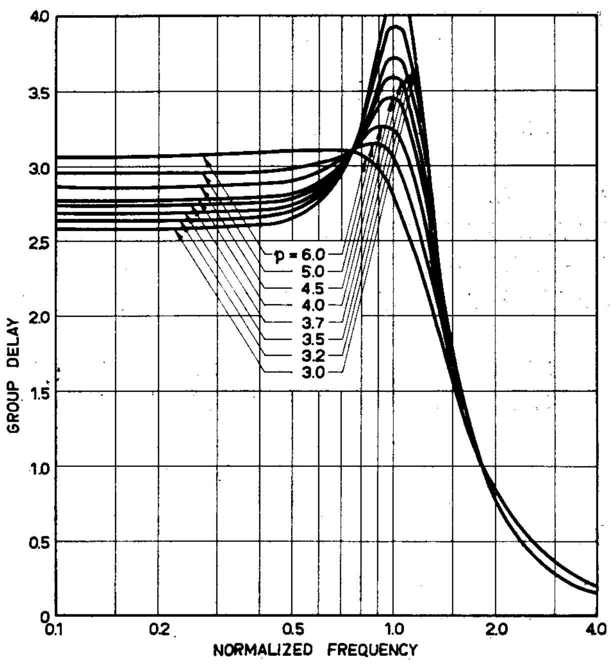

Fig. 5 — Group delays of the 4th order filters

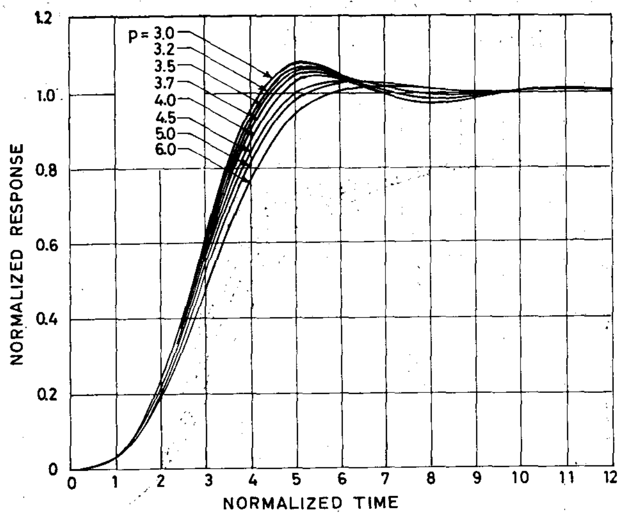

Fig. 6 — Step responses of the 4th order filters.

TABLE II

FOURTH-ORDER FILTERS

p	3.0	3.2	3.5	3.7	4.0	4.5	5.0	6.0
s_1, s_2	-0.8329 $\pm j\,0.3361$	-0.8302 $\pm j\,0.3324$	-0.8274 $\pm j\,0.3273$	-0.8264 $\pm j\,0.3242$	-0.8257 $\pm j\,0.3200$	-0.8275 $\pm j\,0.3137$	-0.8328 $\pm j\,0.3079$	-0.8561 $\pm j\,0.2952$
s_3, s_6	-0.3254 $\pm j\,1.0648$	-0.3475 $\pm j\,1.0629$	-0.3800 $\pm j\,1.0577$	-0.4010 $\pm j\,1.0528$	-0.4318 $\pm j\,1.0434$	-0.4808 $\pm j\,1.0226$	-0.5264 $\pm j\,0.9957$	-0.6049 $\pm j\,0.9239$
B	1.12	1.09	1.04	1.01	0.96	0.85	0.80	0.72
τ	2.29	2.35	2.43	2.48	2.57	2.71	2.85	3.10
$\gamma\%$	9.45	8.14	6.48	5.56	4.41	3.00	2.07	1.14

Fifth-Order Filters

The transfer function of the fifth-order filter is given by

$$F_5(s) = \frac{105p}{105p + 105ps + (45p + 15)s^2 + (10p + 15)s^3 + (p + 6)s^4 + s^5} \tag{29}$$

Using Eqs. 14, 15 and 20, the magnitude and group delay functions are obtained in the following forms

$$A_5(\omega) = \frac{105\,p}{\left[\sum_0^5 c_k\,\omega^{2k}\right]^{1/2}} \tag{30}$$

$$D_5(\omega) = 1 - \frac{\omega^8\,(p^2 - 9\,p + \omega^2)}{\sum_0^5 c_k\,\omega^{2k}} \tag{31}$$

where

$$\sum_0^5 c_k\,\omega^{2k} = 11025\,p^2 + (1575\,p^2 - 3150\,p)\,\omega^2 + (135\,p^2 - 540\,p + 225)\,\omega^4 +$$
$$+ (10\,p^2 - 60\,p + 45)\,\omega^6 + (p^2 - 8\,p + 6)\,\omega^8 + \omega^{10}$$

Again, the steady-state and transent responses for different values of p are plotted in Figs. 7—9 and the data on the locations of the poles, the normalized 3 db bandwidth (B), the per cent overshoot (γ) are the normalized rise-time (τ) are summarized in Table III.

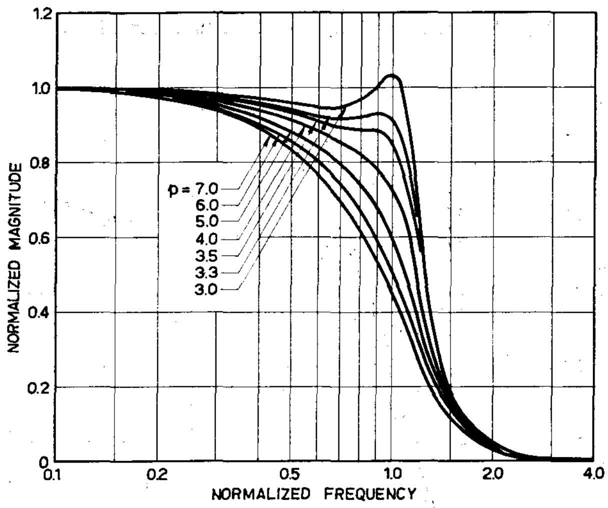

Fig. 7 — Magnitude responses of the 5th order filters

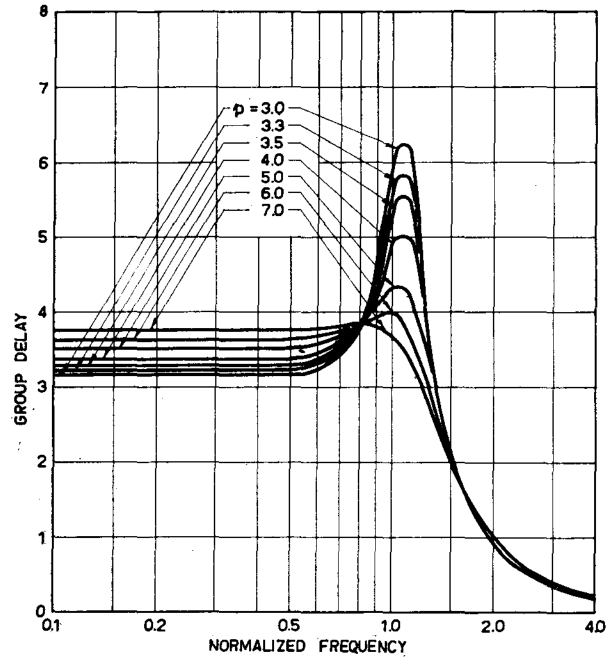

Fig. 8 — Group delays of the 5th order filters

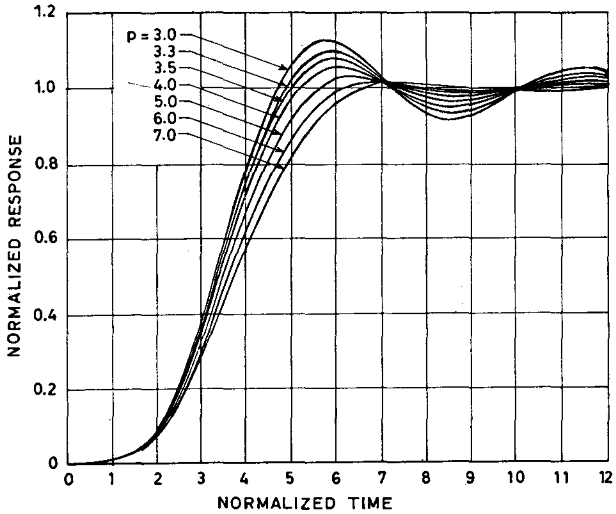

Fig. 9 — Step responses of the 5th order filters

TABLE III

FIFTH-ORDER FILTERS

p	3	3.3	3.5	4.0	5.0	6.0	7.0
s_1	−0.8999	−0.8929	−0.8890	−0.8810	−0.8720	−0.8704	−0.8765
s_2, s_3	−0.7747 ±j 0.5452	−0.7719 ±j 0.5377	−0.7704 ±j 0.5333	−0.7683 ±j 0.5236	−0.7691 ±j 0.5084	−0.7765 ±j 0.4965	−0.7912 ±j 0.4853
s_4, s_5	−0.1994 ±j 1.0947	−0.2254 ±j 1.1021	−0.2427 ±j 1.1056	−0.2851 ±j 1.1099	−0.3665 ±j 1.1022	−0.4414 ±j 1.0759	−0.5070 ±j 1.0331
B	1.18	1.16	1.13	1.02	0.87	0.75	0.68
τ	2.34	2.42	2.46	2.59	2.82	3.04	3.24
γ%	12.56	10.31	9.02	6.43	3.23	1.70	1.06

Sixth-Order Filters

Using the same equations as in the previous cases the transfer function, the magnitude and group delay responses of the sixth-order filters are found as

$$F_6(s) = \frac{945p}{945p - 945ps + (420p+105)s^2 + (105p+105)s^3 + (15p+45)s^4 + (p+10)s^5 + s^6} \tag{32}$$

$$A_6(\omega) = \frac{945p}{\left[\sum_0^6 c_k \omega^{2k}\right]^{1/2}} \tag{33}$$

$$D_6(\omega) = 1 - \frac{\omega^{10}(p^2 - 11p + \omega^2)}{\sum_0^6 c_k \omega^{2k}} \tag{34}$$

where

$$\sum_0^6 c_k \omega^{2k} = 893025p^2 + (99022p^2 - 198050p)\omega^2 + (6300p^2 - 25200p + 11025)\omega^4 +$$
$$+ (315p^2 - 1890p + 1575)\omega^6 + (15p^2 - 120p + 135)\omega^8 + (p^2 - 10p + 10)\omega^{10} + \omega^{12}$$

The magnitude, group delay and transient responses for different values of p are shown in Figs. 10, 11 and 12. For convenience, the pole positions, the normalized rise-time (τ) the normalized 3 db bandwidth (B) and the per cent overshoot (γ) are tabulated in Table IV.

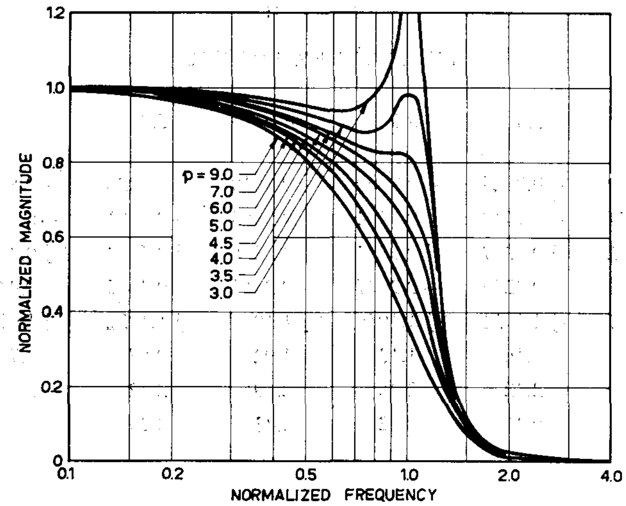

Fig. 10 — Magnitude responses of the 6th order filters

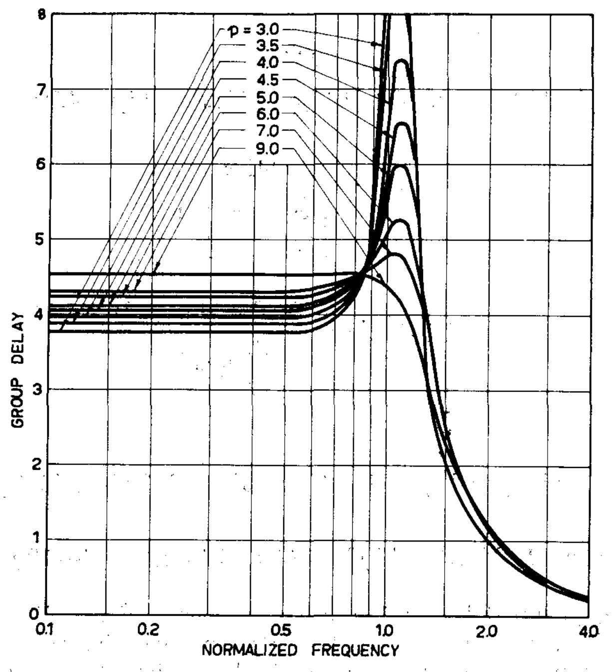

Fig. 11 — Group delays of the 6th order filters

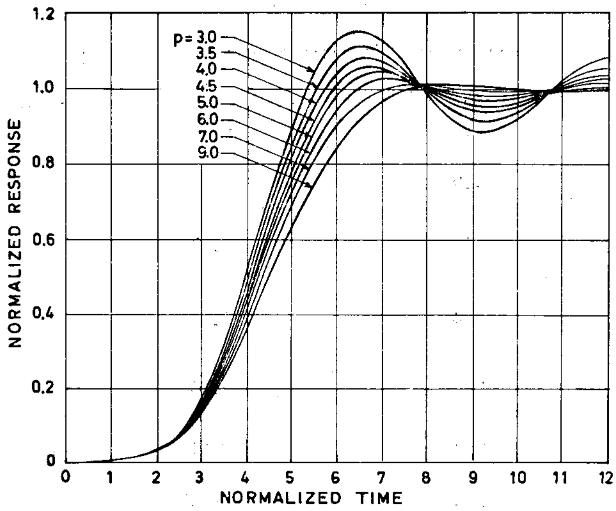

Fig. 12 — Step responses of the 6th order filters

TABLE IV

SIXTH-ORDER FILTERS

p	0.0	3.5	4.0	4.5	5.0	6.0	6.0	9.0
s_1, s_2	− 0.8941 ± j 0.2265	− 0.8838 ± j 0.2219	− 0.8759 ± j 0.2181	− 0.8698 ± j 0.2148	− 0.8652 ± j 0.2121	− 0.8597 ± j 0.2076	− 0.8584 ± j 0.2041	− 0.8689 ± j 0.1980
s_3, s_4	− 0.7120 ± j 0.6855	− 0.7099 ± j 0.6724	− 0.7090 ± j 0.6618	− 0.7090 ± j 0.6530	− 0.7099 ± j 0.6455	− 0.7142 ± j 0.6334	− 0.7219 ± j 0.6234	− 0.7504 ± j 0.6040
s_5, s_6	− 0.1217 ± j 1.0902	− 0.1550 ± j 1.1116	− 0.1887 ± j 1.1266	− 0.2224 ± j 1.1364	− 0.2558 ± j 1.1416	− 0.3206 ± j 1.1416	− 0.3815 ± j 1.1253	− 0.4834 ± j 1.0599
B	1.20	1.19	1.14	1.02	0.89	0.75	0.69	0.62
τ	2.43	2.54	2.65	2.76	2.86	3.07	3.26	3.56
γ %	14.79	10.86	7.91	5.71	4.10	2.10	1.15	0.65

Discussion of the Results and Comparison with Other Systems

It can be seen from the foregoing description that the proposed class of function enables a large variety of filter specifications to be met in practical design, both with regard to the transient response and steady-state characteristics. In this respect these filters are similar to the so-called transitional Butterworth-Thomson filters,but unlike the latter, they provide an improvement over the Butterworth and other types of filters especially when simultaneous

TABLE V

COMPARISON OF THIRD-ORDER FILTERS

Type of Filter	Normalized Rise Time τ	Percent Overshoot $\gamma\,\%$	Normalized Bandwidth $\omega_{3\,db}$
MFM Filters	2.29	8.15	1.00
TBT Filters (m = 0.4)	2.59	3.87	0.86
TBT Filters (m = 0.6)	2.74	2.45	0.81
C Filters (b = 2)	2.30	7.24	0.99
C Filters b = 2.5)	2.34	5.61	0.98
QB Filters (p = 2.5)	2.15	8.37	1.10
QB Filters (p = 3)	2.33	5.21	1.00
QB Filters (p = 3.5)	2.53	3.15	0.93

TABLE VI

COMPARISON OF FIFTH-ORDER FILTERS

Type of Filter	Normalized Rise Time τ	Percent Overshoot $\gamma\,\%$	Normalized Bandwidth $\omega_{3\,db}$
MFM Filters	2.56	12.8	1.00
TBT Filters (m = 0.4)	2.94	5.71	0.82
TBT Filters (m = 0.6)	3.15	3.46	0.74
C Filters (b = 2)	2.64	8.78	0.97
C Filters (b = 2.5)	2.73	5.11	0.92
QB Filters (p = 3.5)	2.46	9.02	1.13
QB Filters (p = 4)	2.59	6.43	1.02
QB Filters (p = 5)	2.82	3.23	0.87

stipulations on the transient response and the bandwidth of the steady-state magnitude characteristic are made. In tables V and VI a few important parameters of interest in the transient and steady-state responses for 3 th and 5 th order Butterworth (MFM), Transitional Butterworth-Thomson (TBT), for $m=0.4$ and $m=0.6$, filters with catenary distribution of poles (C) for $b=2$ and $b=2.5$ and the described class of filters, which will be referred to as quasi-Bessel filters (QB), are compared.

It can be seen from Table VI that the quasi-Bessel filter of the fifth order with p = 4 when compared with the Butterworth filter on the basis of equal rise-time and 3 db bandwidth yields the percent overshoot which is just one half of that for the Butterworth filter. These filters also offer substantial improvement in the transient resposes over the transitional Butterworth-Thomson filters and to a smaller degree, over the filters with catenary distribution of poles which can be noticed from the results summarized in tables V and VI.

In those applications where small transient overshoot of only 1—2 percent can be tolerated, the quasi-Bessel filters yield in some cases virtually the same results as the filters with elliptic distribution of poles proposed by Scanlan, while in some other cases they are slightly inferior to them. For example, the third order filter with elliptic distribution of poles and with the eccentricity ratio a = 0.4 has the normalized rise time $\tau=2.31$ with the transient overshoot of only 1.1 %. However, this overshoot is followed by an undershoot of 3.2 percent. In order to reduce the undershoot to a values of less than 1 %, the eccentricity ratio must be increased to a = 0. 6. For this value of a the per cent overshoot is equal 2.22 and the normalized rise-time is $\tau=2.56$. These results are quite similar to those obtained with the third order quasi-Bessel filter having p = 3. 5. In this connection some very interesting results which have been obtained by Jess and Schuessler [12] should be mentioned. They have studied the problem of optimizing the transient response of pulse networks by means of analog and digital computers and found that, if ringing in time response can be tolerated, the rise-time is shortest if the step response approximates its final value in a Chebyshev sense.

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