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ON THE »RECTANGLE FUNCTIONAL EQUATION« AND THE FUNCTIONAL EQUATION $|f(x+y)-f(x-y)|=|f(x+\bar{y})-f(x-\bar{y})|$ CONNECTED WITH IVORY'S THEOREM*

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§ 1. Introduction

J. Aczél considered the following functional equation:

$$f(x+u,\ y+v)+f(x-u,\ y+v)+f(x-u,\ y-v)+f(x+u,\ y-v)=4f(x,\ y), \tag{1}$$

where $f(x, y)$ is a one-valued real function of two real variables x, y.

He obtained the following interesting theorem:

If $f(x, y)$ is measurable in the whole xy-plane, then the solution of (1) is the following and only this:

$$f(x,\ y)=axy+bx+cy+d,$$

where a, b, c, d are arbitrary real constants.

Suppose that $ABCD$ is an arbitrary rectangle whose sides are parallel to the coordinate axes in the whole xy-plane. Geometrically speaking, (1) means that the arithmetic mean of the four values of $f(x, y)$ at the four vertices of $ABCD$ is equal to the value of $f(x, y)$ at the center of this rectangle. If we represent every point in the whole xy-plane by a complex number, (1) is the following form:

$$f(x+y)+f(x-\bar{y})+f(x-y)+f(x+\bar{y})=4f(x), \tag{2}$$

where x, y are complex variables.

Now, we assume that $f(z)$ is a one-valued complex function of a complex variable z.

By (2) we have the following functional equation:

$$|f(x+y)+f(x-y)-2f(x)|=|f(x+\bar{y})+f(x-\bar{y})-2f(x)|. \tag{3}$$

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* Presented November 20, 1966 by D. S. Mitrinović and P. M. Vasić.

Next, we shall consider the functional equation (3) from another point of view.

In previous papers (see [3] or [4] or [5]) we solved the following functional equation connected with Ivory's Theorem

$$(4) \qquad |f(x+y)-f(x-y)| = |f(x+\bar{y})-f(x-\bar{y})|,$$

where $f(z)$ is an entire function of z and x, y are complex variables.

We shall prove that (4) implies (3) if $f(z)$ is an entire function of z.

Lemma 1. *Suppose that $f(z)$, $g(z)$ are entire functions of z. If $|f'(z)| = |g'(z)|$ holds in $|z| < +\infty$ and $f(0)=g(0)=0$ holds, then $|f(z)| = |g(z)|$ holds in $|z| < +\infty$.*

Proof. Since $f(z)$, $g(z)$ are entire functions of z and $|f'(z)| = |g'(z)|$ holds in $|z| < +\infty$, we have $f'(z) = e^{i\theta} g'(z)$ in $|z| < +\infty$ where θ is a real constant. Hence we have $f(z) = e^{i\theta} g(z) + C$ in $|z| < +\infty$ where C is a complex constant. So, by the assumption $f(0) = g(0) = 0$ we have $C = 0$. Hence we have $f(z) = e^{i\theta} g(z)$ and so $|f(z)| = |g(z)|$ in $|z| < +\infty$. Q.E.D.

Lemma 2. *If $f(z)$ is an entire function of z, then $\Delta |f(z)|^2 = 4|f'(z)|^2$ where Δ stands for the Laplacian $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ ($z = x+iy$, $i=\sqrt{-1}$, x, y real).*

Proof. Since this lemma is familiar, we omit the proof. (See [2].)

By (4) we have

$$(5) \qquad |f(x+y)-f(x-y)|^2 = |f(x+\bar{y})-f(x-\bar{y})|^2.$$

Taking the Laplacian $\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}$ of both sides of (5) with respect to x ($x = s+it$, $i = \sqrt{-1}$, s, t real), by Lemma 2 we have

$$4|f'(x+y)-f'(x-y)|^2 = 4|f'(x+\bar{y})-f'(x-\bar{y})|^2,$$

or

$$(6) \qquad |f'(x+y)-f'(x-y)| = |\overline{f'(x+\bar{y})}-\overline{f'(x-\bar{y})}|.$$

When x is arbitrarily fixed,

$$f(x+y)+f(x-y)-2f(x) \text{ and } \overline{f(x+\bar{y})}+\overline{f(x-\bar{y})}-\overline{2f(x)}$$

are entire functions of y with

$$(f(x+y)+f(x-y)-2f(x))_{y=0} = (\overline{f(x+\bar{y})}+\overline{f(x-\bar{y})}-\overline{2f(x)})_{y=0} = 0$$

and by (6) we have

$$\left| \frac{\partial}{\partial y}(f(x+y)+f(x-y)-2f(x)) \right| = \left| \frac{\partial}{\partial y}(\overline{f(x+\bar{y})}+\overline{f(x-\bar{y})}-\overline{2f(x)}) \right|.$$

Hence, by Lemma 1 we have

$$|f(x+y)+f(x-y)-2f(x)|=|\overline{f(x+y)}+\overline{f(x-y)}-2\overline{f(x)}|,$$

and so

$$|f(x+y)+f(x-y)-2f(x)|=|f(x+\bar{y})+f(x-\bar{y})-2f(x)|.$$

Hence (4) implies (3) if $f(z)$ is an entire function of z.

So, if (2) or (4) holds, then (3) holds. But the converse is not true as the example $f(z)=z^3$ shows.

In this paper, by Lemma 2 and a previous result (see [3] or [4] or [5]) we shall solve the functional equation (3) and shall state applications of this result to the theory of functional equations.

§ 2. Solution of the functional equation (3)

Theorem. *If $f(z)$ is an entire function of z, then the solutions of* (3) *are the following and only these:*

$$f(z)=a\sin\alpha z+b\cos\alpha z+cz+d,$$

or

$$f(z)=a\sinh\alpha z+b\cosh\alpha z+cz+d,$$

or

$$f(z)=az^3+bz^2+cz+d,$$

where a, b, c, d are arbitrary complex constants and α is an arbitrary real constant.

Proof. By (3) we hawe

$$|f(x+y)+f(x-y)-2f(x)|^2=|\overline{f(x+\bar{y})}+\overline{f(x-\bar{y})}-2\overline{f(x)}|^2. \tag{7}$$

Taking the Laplacian $\frac{\partial^2}{\partial s^2}+\frac{\partial^2}{\partial t^2}$ of both sides of (7) with respect to y $(y=s+it,\ s,\ t$ real), by Lemma 2 in § 1 we have ($\overline{f(\bar{z})}$ is an entire function of z)

$$4|f'(x+y)-f'(x-y)|^2=4|\overline{f'(x+\bar{y})}-\overline{f'(x-\bar{y})}|^2.$$

Hence we have

$$|f'(x+y)-f'(x-y)|=|f'(x+\bar{y})-f'(x-\bar{y})|. \tag{8}$$

Putting $F(z)=f'(z)$, by (8) we have

$$|F(x+y)-F(x-y)|=|F(x+\bar{y})-F(x-\bar{y})|. \tag{9}$$

So, by (9) and a previous result (see [3] or [4] or [5]) we have

$$F(z)=a\sin\alpha z+b\cos\alpha z+c,$$

or

$$F(z)=a\sinh\alpha z+b\cosh\alpha z+c,$$

or

$$F(z)=az^2+bz+c,$$

where a, b, c are complex constants and α is a real constant. Hence we have three functions in Theorem.

Direct substitution shows that our equation (3) is actually satisfied by these three functions.

Thus the theorem is proved.

§ 3. Applications of the result in § 2 to the theory of functional equations

By the above Theorem we can systematically solve the following functional equations (see [1]) under the hypothesis that $f(z)$, $g(z)$, $h(z)$ are entire functions of z and x, y are complex variables:

(i) $$f(x+y)+f(x-\bar{y})+f(x-y)+f(x+\bar{y})=4f(x)$$

(the "rectangle functional equation"),

(ii) $$f(x+y)=f(x)+f(y),$$

(iii) $$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2},$$

(iv) $$f(x+y)=f(x)f(y),$$

where $f(z)$ is real on the real axis,

(v) $$f(x+y)+f(x-y)=2f(x)+2g(x)h(y),$$

where $h(z)$ is real on the real axis,

(vi) $$f(x+y)+f(x-y)=2f(x)+2g(y),$$

where $g(z)$ is real on the real axis,

(vii) $$f(x+y)+f(x-y)=2f(x)+2f(y),$$

where $f(z)$ is real on the real axis,

(viii) $$f(x+y)+f(x-y)=2f(x)g(y)$$

where $g(z)$ is real on the real axis,

(ix) $$f(x+y)+f(x-y)=2f(x)f(y),$$

where $f(z)$ is real on the real axis,

(x) $$f(x+y)+f(x-y)=2f(x)\cos ay,$$

where a is a real constant,

(xi) $$f(x+y)+f(x-y)=2f(x)\cosh ay,$$

where a is a real consant.

SOLUTION OF (i). If $f(z)$ satisfies (i), then $f(z)$ satisfies (3). Hence, selecting from the solutions of (3) in the Theorem that function which satisfies (i), we have $f(z)=az+b$ where a, b are arbitrary complex constants. There are no other solutions of (i) which are entire functions, but (see Introduction) $f(z)=A(z^2-\bar{z}^2)+Bz+C\bar{z}+D$ where A, B, C, D are arbitrary is the only measurable solution in $|z|<+\infty$.

Solution of (ii). If $f(z)$ satisfies (ii), then $f(z)$ satisfies (3). Hence, selecting from the solutions of (3) in the Theorem that function which satisfies (ii), we have $f(z)=az$ where a is an arbitrary complex constant. The solution of (ii) is only this.

Solution of (iii). If $f(z)$ satisfies (iii), then $f(z)$ satisfies (3). Hence, selecting from the solutions of (3) in the Theorem that function which satisfies (iii), we have $f(z)=az+b$ where a, b are arbitrary complex constants. The solution of (iii) is only this.

Solution of (iv). Since $f(z)$ is real on the real axis, we have $f(\bar{y})=\overline{f(y)}$. By (iv) we have

$$|f(x+y)+f(x-y)-2f(x)|=|f(x)||f(y)+f(-y)-2|,$$
$$|f(x+\bar{y})+f(x-\bar{y})-2f(x)|=|f(x)||f(\bar{y})+f(-\bar{y})-2|.$$

Hence $f(z)$ satisfies (3). Hence, selecting from the solutions of (3) in the Theorem those functions which satisfy (iv), we have $f(z)=0$ or $f(z)=e^{az}$ where a is an arbitrary real constant. The solutions of (iv) are only these.

Solution of (v). Since $h(z)$ is real on the real axis, we have $h(\bar{y})=\overline{h(y)}$. By (v) we have

$$|f(x+y)+f(x-y)-2f(x)|=2|g(x)||h(y)|,$$
$$|f(x+\bar{y})+f(x-\bar{y})-2f(x)|=2|g(x)||h(\bar{y})|.$$

Hence $f(z)$ satisfies (3). Hence, selecting from the solutions of (3) in the Theorem that function which satisfies (v), we have $f(z)=az^3+bz^2+cz+d$ where a, b, c, d are arbitrary complex constants. The solution $f(z)$ of (v) is only this. $g(z)$, $h(z)$ are easily determined.

Solution of (vi). This is the special case of (v). Hence, selecting from the solutions of (v) those functions which satisfy (vi), we have $f(z)=\alpha z^2+$ $+bz+c$, $g(z)=\alpha z^2$ where α is an arbitrary real constant and b, c are arbitrary complex constants. The solution of (vi) is only this.

Solution of (vii). This is the special case of (vi). Hence, selecting from the solutions of (vi) that function which satisfies (vii), we have $f(z)=\alpha z^2$ where α is an arbitrary real constant. The solution of (vii) is only this.

Solution of (viii). Since $g(z)$ is real on the real axis, we have $g(\bar{y})=\overline{g(y)}$. By (viii) we have

$$|f(x+y)+f(x-y)-2f(x)|=2|f(x)||g(y)-1|,$$
$$|f(x+\bar{y})+f(x-\bar{y})-2f(x)|=2|f(x)||g(\bar{y})-1|.$$

Hence $f(z)$ satisfies (3). Hence, selecting from the solutions of (3) in the Theorem those functions which satisfy (viii), we have

$$f(z) = a \sin \alpha z + b \cos \alpha z,$$

or

$$f(z) = a \sinh \alpha z + b \cosh \alpha z,$$

or

$$f(z) = az + b,$$

where a, b are arbitrary complex constants and α is an arbitrary real constant. The solutions $f(z)$ of (viii) are only these. In each case $g(z)$ is easily determined.

SOLUTION OF (ix). This is the special case of (viii). Hence, selecting from the solutions of (viii) those functions which satisfy (ix), we have

$$f(z) = 0,$$

or

$$f(z) = \cos \alpha z,$$

or

$$f(z) = \cosh \alpha z,$$

where α is an arbitrary real constant. The solutions of (ix) are only these.

SOLUTION OF (x). This is the special case of (viii). Hence, selecting from the solutions of (viii) those functions which satisfy (x), we have

$$f(z) = a \sin \alpha z + b \cos \alpha z \qquad (\text{when } \alpha \neq 0),$$

$$f(z) = az + b \qquad (\text{when } \alpha = 0),$$

where a, b are arbitrary complex constants. In each case the solution of (x) is only this.

SOLUTION OF (xi). This is the special case of (viii). Hence, selecting from the solutions of (viii) those functions which satisfy (xi), we have

$$f(z) = a \sinh \alpha z + b \cosh \alpha z \qquad (\text{when } \alpha \neq 0),$$

$$f(z) = az + b \qquad (\text{when } \alpha = 0),$$

where a, b are arbitrary complex constants. In each case the solution of (xi) is only this.

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