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ON THE PAPERS № 123 — № 127 OF L. KARADŽIĆ

Julian Musielak

The author L. Karadžić[1] formulates a number of theorems giving necessary and sufficient conditions of convergence resp. uniform convergence of numerical series resp. trigonometric, power and Dirichlet series. Unfortunately, all the theorems are false in their necessity parts, and not all proofs of sufficiency are clear enough. As regards the necessity parts I shall provide counter-examples, but it is well-known to everyone working in series that no necessary conditions of convergence of series can be given by means of order of growth of their terms.

I shall analyse Theorem 1 in [1] more in detail, for this theorem is the base of all further author's considerations and also the source of all mistakes. The author states that

$$\text{(a)}\quad \lim_{n\to\infty}\prod_{k=1}^{n}\frac{1-b_k}{1-a_k}=A\,(0<A<\infty),\qquad \text{(b)}\quad \sum_{n=1}^{\infty} a_n^2+b_n^2<\infty,$$

$$\text{(c)}\quad \alpha_n-\beta_{n+r}=O(a_n)\qquad (r=0,\pm 1,\ldots,\pm s)$$

are necessary and sufficient conditions for the convergece of series

$$(*)\quad \sum_{n=1}^{\infty}\alpha_n a_n-\beta_n b_n\qquad (\alpha_n>0,\ \beta_n>0,\ 0<a_n<1,\ 0<b_n<1).$$

(In the paper [1] the conditions (a), (b), (c), (*) have the numbers (10), (11), (12), (8), respectively).

The condition (c) is not quite clear but it follows from authors proofs that should be read as $|\alpha_n-\beta_{n+r}|\leqslant Ka_n$ for $r=0,\pm 1,\ldots,\pm s$, with K independent of n and of s. This means, that (c) can be written as

$$\text{(c')}\quad \left|\frac{\alpha_n-\beta_{n+r}}{a_n}\right|\leqslant K \text{ for arbitrary } n \text{ and } r=0,\pm 1,\ldots,$$

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[1] *Publikacije Elektrotehničkog fakulteta Univerziteta u Beogradu*, Serija: *Matematika i Fizika*, № 123—127, (1964):

[1] Remarques sur certains théorèmes de la théorie de séries (№ 123);

[2] Sur la sommabilité d'une classe de séries selon un procédé déterminé (№ 124);

[3] Un théorème relatif aux séries trigonométriques (№ 125);

[4] Un théorème sur la convergence uniforme de la série de Taylor à la bordure (№ 126);

[5] Un théorème sur la convergence uniforme de la série de Dirichlet à la bordure (№ 127).

with K independent of n and of r. Writing $r=-n+1$ we get $|\alpha_n-\beta_1|\leqslant Ka_n$; hence α_n/a_n is bounded, say by K'. Now, puting $r=0$ in (c′) we get $|\alpha_n-\beta_n|\leqslant Ka_n$, and so β_n/a_n is also bounded, say by K''. Hence, by the assumption (b),

$$\sum_{n=1}^{\infty}|\alpha_n a_n-\beta_n b_n|\leqslant \sum_{n=1}^{\infty}\alpha_n a_n+\sum_{n=1}^{\infty}\beta_n b_n\leqslant K'\sum_{n=1}^{\infty}a_n^2+K''\sqrt{\sum_{n=1}^{\infty}a_n^2}\sqrt{\sum_{n=1}^{\infty}b_n^2}<\infty,$$

and convergence of series (*) is proved very easy, without assumption (a). But conditions (b) and (c′) do not imply (a) — as an example one can take $a_n=1/n$, $b_n=1/n^2$, $\alpha_n=\beta_n=1$, where (b) and (c′) are satisfied, but (a) is not, for

$$\prod_{k=1}^{n}(1-b_k)/(1-a_k)=\prod_{k=1}^{n}(1+1/k)\to\infty.$$

Hence (a), (b) and (c) are not necessary for (*). This will be shown directly by a counter-example later.

It is easy to find the mistake in author's proof of Theorem 1. The author majorizes the sum $\left|\sum_{k=n+1}^{n+p}\alpha_k a_k-\beta_k b_k\right|$ by meas of three terms, say $I_1+I_2+I_3$. Than he says that (a), (b), (c) are the respective necessary and sufficient conditions for convergence of I_1, I_2, I_3. Hence he concludes that the conditions (a), (b), (c) together give the necessary and sufficient condition for the sum $I_1+I_2+I_3$.

Now I shall provide counter-examples for the necessity parts of all theorems.

[1] Theorem 1. $a_n=3/4$, $b_n=1/2$, $\alpha_n=n/2$, $\beta_n=3\,n/4$; the series (8) is convergent, but none of the conditions (10)—(12) is satisfied.

Theorem 2. The same example; (13) and (15) are divergent.

Theorem 3. $a_n=b_n=1/2$.

Theorem 4. $S_n=T_n=n$.

Theorem 5. $a_n=(-1)^n/\sqrt{n}$.

[2] Theorem 1. $a_n=b_n=1$, $u_n(t)=\left(\frac{2}{\pi}\arctan t\right)^{n-1}$.

Theorem 2. $a_n=(-1)^{n-1}/\sqrt{n}$, $u_n(t)$ as above.

[3] Theorems on pages 17, 18 and 19. $a_n=b_n=1/\sqrt{n}$ (this is a textbook example of Foorier series uniformly convergent in each subinterval (a,b) such that $0<a<b<2\pi$).

[4] Theorem on page 20. $\alpha=0$, $\delta=\pi/2$, $a_n=1/\sqrt{n}$.

[5] Theorem on page 23. $\lambda_n=n$, $\delta=\pi/2$, $a_n=1/\sqrt{n}$, $C=0$.