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ON THE SENSITIVITY OF LINEAR CONTINUOUS SYSTEMS*

Milić R. Stojić

Summary: This paper, based upon the application of algebraic concept for the analysis and synthesis of linear dynamical systems, presents an analytical procedure for the sensitivity analysis of linear continuous systems. The procedure applies to the sensitivity investigation of system transfer function poles and zeros due to the small changes of system parameters. The suitable defined sensitivities and application of the Chebyshev functions enable the aforementioned sensitivity analysis and other related problems to be advantageously performed by the proposed procedure.

An important problem of the analysis and design of linear networks and control systems is the influence of certain parameter variations on the system performance. This problem could be indetified as a „sensitivity problem". There are many factors in the design of linear dynamical systems which cause the sensitivity problem to arise. For example, in the design of linear control systems it is often necessary to suppose that the values of system parameters do not deviate from the data on which the design is based. Since it is not always possible, during the application of corresponding dynamical system, to maintain the nominal values of system parameters unchanged, the validity of the above assumption should be checked in order to obtain the information about the possible modification and improvement of system performance. Moreover, such a sensitivity analysis should answer the question whether the influence of some parameter variations can be neglected or not. Thus, the sensitivity analysis may be directed toward making a new aspect of control system synthesis. The same hold for adaptive control since the sensitivity analysis can give highly valuable information about the controllable parameters on which the additional adaptation loop should act in order to make the system performance invariant despite the changes in uncontrollable parameters.

In general, two approaches to the sensitivity analysis of linear continuous systems have been done, both investigating the effects of certain parameters variations on the system transfer function. The first approach is based upon the application of the Bode definition of sensitivity function [1], which is given as the logarithmic derivative of the system fransfer function with respect to the logarithm of the gain constant. The socond approach uses the definition of so called pole-zero sensitivity [2—5] which is more convenient in the sense of practical application, since the network and control system synthesis most frequently is carried out by investigating of the pole and zero locations of system transfer function.

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This paper presents an analytical procedure of sensitivity analysis of linear continuous systems, which is based upon the application of the Chebyshev functions and algebraic concept of analysis and synthesis of feedback control systems [7]. The idea was first introduced by Mitrović who investigated the differential changes in the relative damping coefficient and undamped natural frequency of the roots of characteristic equation due to the differential changes in parameters of linear continuous systems [4]. Using generalized Mitrović's method and the more convenient definition of sensitivity, it has been shown [5] that the sensitivity problem in linear continuous and sampled-data control systems can be solved in a general but extremely effective manner. The proposed procedure can be applied to any system configuration and locations of system parameters and therefore it appears to be a general solution of the sensitivity problem in linear control systems.

As the following will show, the proposed procedure in this paper guides to the rather simplified definition of sensitivity and to the useful application of the Chebyshev functions. Since the Chebyshev functions as well as their derivatives were tabulated [7] and may be conveniently programmed on both analog and digital computers, the proposed analytical procedure is extremely simplified in the sense of practical application. Furthermore, by applying this procedure, the relative stability of self-excited nonlinear oscillations and sensitivity of sustained oscillations in nonlinear control systems as well as the sensitivity of discrete control systems and other related problems have been solved in a straightforward manner [8], [9]. The procedure may be applied to any system configuration whenever it is required to evaluate the effects that the small variations of any system parameters have on the corresponding location of system transfer function poles and zeros.

DEFINITION OF SENSITIVITY

It is often necessary to express the transfer function of a linear dynamical system as a ratio of two polinomials in complex variable s. In these cases, the investigation of effects that the small variations in system parameters have on the poles and zeros of the system transfer function can be reduced to the investigation of the root sensitivity of the algebraic equation

$$\sum_{k=0}^{n} a_k s^k = 0 \tag{1}$$

where the coefficients a_k $(k=0, 1, \ldots, n)$ are determined by the values of system parameters q_r $(r=1, 2, \ldots, m)$, namely

$$a_k = a_k(q_1, q_2, \ldots, q_m) \qquad (k=0, 1, \ldots, n) \tag{2}$$

For a pertinent set of the system parameter values q_r^0 $(r=1, 2, \ldots, m)$, by means of Eq. 2, the corresponding numerical values of the coefficients a_k^0 $(k=0, 1, \ldots, n)$ which determine the root location of the algebraic equation (1), can be obtained. Evidently, the parameter variatons will cause the corresponding perturbation in the roots location in Eq. 1. In order to evaluate the degree of correspondence between the system parameters variations and the root location, it is convenient to define the complex root sensitivity and real root sensitivity, separately, and to focus the attention on each individual root indenpendently.

In order to define the complex root sensitivity, let us express the i-th pair of complex roots $s_{i,i+1}$ of Eq. 1 as

$$s_{i,i+1} = -(\omega_n)_i \, \zeta_i \pm j \, (\omega_n)_i \sqrt{1-\zeta_i^2} \tag{3}$$

where ζ_i is the relative damping coefficient and $(\omega_n)_i$ is the undamped natural frequency of the corresponding pair of complex roots (in the further developments the subscript „n" will be omitted).

The complex root sensitivity may be conveniently defined by means of „complex root sensitivity matrix" which is introduced as

$$S_{i,r} = (S_{i,r}^{\zeta}, \; S_{i,r}^{\omega}) = \begin{Vmatrix} S_{i,r}^{\zeta} \\ \\ S_{i,r}^{\omega} \end{Vmatrix} \tag{4}$$

The complex root sensitivity matrix elements $S_{i,r}^{\zeta}$ and $S_{i,r}^{\omega}$ are respectively: the relative damping coefficient sensitivity and natural frequency sensitivity of complex roots $s_{i,i+1}$ with respect to the variation of a parameter q_r. These sensitivities have been defined as the logarithmic derivatives of relative damping coefficient ζ_i and undamped natural frequency ω_i with respect to the logarithm of a system parameter q_r, i. e.

$$S_{i,r}^{\zeta} = \frac{\partial \ln \varsigma_i}{\partial \ln q_r}, \qquad S_{i,r}^{\omega} = \frac{\partial \ln \omega_i}{\partial \ln q_r}$$

In like manner, the sensitivity of the j-th real root of algebraic equation 1 with respect to the variation of system parameter q_r has been defined as

$$S_{j,r}^{\sigma} = \frac{\partial \ln \sigma_j}{\partial \ln q_r} \tag{5}$$

As it will be shown later, both the complex root sensitivity matrix and real root sensitivity may be readily evaluated using the Chebyshev functions and algebraic concept of control system analysis and synthesis [7].

COMPLEX ROOT SENSITIVITY MATRIX

To develop an expression for the complex root sensitivity matrix, it is necessary to express the complex variable s in Eq. 1 as

$$s = \omega \, e^{\pm j\theta} = -\omega \, \zeta \pm j \, \omega \sqrt{1-\zeta^2}, \qquad (\zeta = -\cos\theta) \tag{6}$$

Substituting Eq. 6 into Eq. 1 and then applying the condition that the real and imaginary parts of Eq. 1 must go to zero indenpendently, the Eq. 1 may be rewritten as two simultaneous equations

$$\begin{gathered}\sum_{k=0}^{n}(-1)^k a_k \omega^k T_k(\zeta)=0 \\ \sqrt{1-\zeta^2}\sum_{k=0}^{n}(-1)^k a_k \omega^k U_k(\zeta)=0\end{gathered} \tag{7}$$

where

$$T_k(\zeta)=\cos(k\arccos\zeta),\ U_k(\zeta)=\frac{\sin(k\arccos\zeta)}{\sqrt{1-\zeta^2}} \tag{8}$$

are the Chebyshev functions of first and second kind, respectively, with the argument $0\leqslant\zeta\leqslant 1$. These functions can be obtained by the well-known reccurence formulae

$$T_{k+1}(\zeta)-2\zeta\,T_k(\zeta)+T_{k-1}(\zeta)=0,\quad U_{k+1}(\zeta)-2\zeta\,U_k(\zeta)+U_{k-1}(\zeta)=0 \tag{9}$$

with $T_0(\zeta)=1,\ \ T_1(\zeta)=\zeta,\ \ U_0(\zeta)=0,$ and $U_1(\zeta)=1$. The derivatives

$$\frac{\mathrm{d}T_k(\zeta)}{\mathrm{d}\zeta}=k\,U_k(\zeta),\qquad \frac{\mathrm{d}[\sqrt{1-\zeta^2}\,U_k(\zeta)]}{\mathrm{d}\zeta}=-\frac{k}{\sqrt{1-\zeta^2}}T_k(\zeta) \tag{10}$$

play an important role in the developments that follow.

Owing to Eqs. 2, Eqs. 7 represent implicit relationship between the relative damping coefficient, undamped natural frequency, and system parameters. Thus, to obtain the complex root sensitivity matrix, it is first necessary to differenciate Eqs. 7 with respect to the parameter q_r. In doing so and applying the relationships 10, with simple manipulations it can be obtained

$$\begin{aligned}&\frac{\partial\ln\zeta}{\partial\ln q_r}\zeta\sum_{k=1}^{n}(-1)^k ka_k\omega^k U_k(\zeta)+\frac{\partial\ln\omega}{\partial\ln q_r}\sum_{k=1}^{n}(-1)^k ka_k\omega^k T_k(\zeta)\\ &=-q_r\sum_{k=0}^{n}(-1)^k\frac{\partial a_k}{\partial q_r}\omega^k T_k(\zeta),\\ &-\frac{\partial\ln\zeta}{\partial\ln q_r}\frac{\zeta}{\sqrt{1-\zeta^2}}\sum_{k=1}^{n}(-1)^k ka_k\omega^k T_k(\zeta)+\frac{\partial\ln\omega}{\partial\ln q_r}\sum_{k=1}^{n}(-1)^k ka_k\omega^k U_k(\zeta)\\ &=-q_r\sum_{k=0}^{n}(-1)^k\frac{\partial a_k}{\partial q_r}\omega^k U_k(\zeta)\end{aligned} \tag{11}$$

Now, if the numerical values a_k^0 $(k=0,\ 1,\ldots,\ n)$ corresponding to a specific set of parameters q_r^0 $(r=1,\ 2,\ldots,\ m)$ are given and the attention is focused on the i-th pair of complex roots, by sybstituting the numerical values a_k^0 $(k=0,\ 1,\ldots,\ n)$, ζ_i, and ω_i into Eqs. 11, one can evaluate the complex root sensitivity matrix in the form

$$S_{i,\,r}=-A_i^{-1}\,Q_i^r \tag{12}$$

The matrix A is a square matrix 2×2 having the elements

$$(13)\qquad \begin{aligned} A_{11} &= \zeta_i \sum_{k=1}^{n} (-1)^k k a_k \omega_i^k U_k(\zeta_i), & A_{22} &= \sum_{k=1}^{n} (-1)^k k a_k \omega_i^k T_k(\zeta_1) \\ A_{21} &= -\frac{\zeta_i}{\sqrt{1-\zeta_i^2}} \sum_{k=1}^{n} (-1)^k k a_k \omega_i^k T_k(\zeta), & A_{12} &= \sum_{k=1}^{n} (-1)^k k a_k \omega_i^k U_k(\zeta_i) \end{aligned}$$

In the case when the i-th pair of complex roots is single, Eqs. 11 are indenpendent on account of the matrix A_i is nonsingular. However, if it is desired to evaluate the complex root sensitivity matrix for a double or multiple i-th pair of complex roots, the matrix A_i would become singular and therefore a rather complicated expressions, than the matrix equation 12, should be used. These cases will not be treated here because in any but carefully chosen examples the control system transfer function contains only single conjugate complex poles and zeros. The same hold for the real root sensitivity, which will be evaluated in the following section.

The matrix Q_i^r is a column matrix 1×2 with the elements

$$(14)\qquad Q_{i,1}^r = q_r^0 \sum_{k=0}^{n} (-1)^k \frac{\partial a_k}{\partial q_r^0} \omega_i^k T_k(\zeta_i), \quad Q_{i,2}^r = q_r^0 \sum_{k=0}^{n} (-1)^k \frac{\partial a_k}{\partial q_r^0} \omega_i^k U_k(\zeta_i)$$

The derivatives $\partial a_k / \partial q_r^0$ $(k=0, 1, \ldots, n)$ appearing in the above expressions of $Q_{i,1}^r$ and $Q_{i,2}^r$ are to be evaluated by differentiating Eqs. with respect to the parameter q_r and then using the nominal numerical values of system parameters q_r^0 $(r=1, 2, \ldots, m)$.

Eqs. 13 indicate that the elements of matrix A_i stay unchanged regarding a variation of any system parameter. Thus, if it is desired to evaluate the i-th pair complex root sensitivity matrix in respect to each of m parameters, it is not necessary to calculate more but one matrix A_i, and only the corresponding matrices Q_i^r should be numerically calculated regarding each parameter. In most general case, when it is desired to calculate the complex root sensitivity matrices of p pairs of complex roots with respect to each of m system parameters, it is necessary to compute p matrices A_i $(i=1, 2, \ldots, p)$ and $m \cdot p$ matrices Q_i^r $(i=1, 2, \ldots, p)$, $(r=1, 2, \ldots, m)$. As it was mentioned earlier, the numerical calculations implied in the proposed procedure is not at all laborious, particulary if computational aids are available.

REAL ROOT SENSITIVITY

According to Eq. 6, the j-th real root (real roots have $\zeta = 1$) of Eq. 1 may be defined with $-\sigma_j$. In this case, the Chebyshev functions of first kind become $T_k(\zeta) \equiv 1$. On the other hand, the second equation of Eqs. 7 for $\zeta = 1$ becomes indentically equal to zero and therefore the real root sensitivity may be achieved by handling only the first equation of Eqs. 7 in the form

$$(15)\qquad \sum_{k=0}^{n} (-1)^k a_k (\sigma_j)^k = 0$$

Since the coefficients a_k $(k = 0, 1, \ldots, n)$ in the above equation are the functions of system parameters, Eq. 15 can be considered as an implicit relationship between the real root $-\sigma_j$ and system parameters q_r $(r = 1, 2, \ldots, m)$. By differentiating this equation with respect to the system parameter q_r, after simple manipulations, the same result for real root sensitivity as that given in reference 5 can be obtained

$$S^{\sigma}_{j,r} = \frac{q_r^0 \sum_{k=0}^{n} (-1)^k \frac{\partial a_k}{\partial q_r^0} (\sigma_j)^k}{\sum_{k=0}^{n} (-1)^k k a_k (\sigma_j)^k} \tag{16}$$

For the given set of numerical values q_r^0 $(r = 1, 2, \ldots, m)$ Eqs. 2 and Eq. 16 place in evidence the procedure of calculation of the real root sensitivity. This procedure as well as the procedure of calculation of the complex root sensitivity matrix will be illustrated by the following example.

EXAMPLE

Consider the single-loop linear control system, shown in fig. 1, which has the following open-loop transfer function

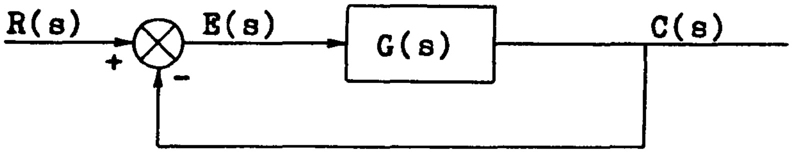

Fig. 1. System block diagram

$$\frac{C}{E}(s) = G(s) = K \frac{s + \exp(-a)}{s(s^2 + 2bcs + c^2)} \tag{17}$$

The corresponding closed-loop transfer function is obtained as

$$\frac{C}{R}(s) = K \frac{s + \exp(-a)}{s^3 + 2bcs^2 + (c^2 + K)s + K\exp(-a)} \tag{18}$$

According to Eq. 18, the system closed-loop transfer function has only one zero $z = -\exp(-a)$ and three poles which are determined by the system characteristic equation

$$s^3 + 2bcs^2 + (c^2 + K)s + K\exp)(-a) = 0 \tag{19}$$

The coefficients a_k of the above characteristic equation are not given in numerical form, but they are the functions of system parameters $q_1 = a$, $q_2 = b$, $q_3 = c$, and $q_4 = K$. For the parameter values $a = 1.45$, $b = 0 \cdot 9$, $c = 0 \cdot 5$, and $K = 0 \cdot 389$ the system closed-loop transfer function has the real zero $z_1 = -0 \cdot 23$, real pole $\sigma_1 = -0 \cdot 18$, and two conjugate complex poles $s_{1,2} = -0 \cdot 36 \pm j\ 0.62$

which are determined by $\zeta_1 = 0.5$ and $\omega_1 = 0.72$. Using the previously defined sensitivities, it is necessary to investigate how the small changes in system parameters a, b, c, and K effect the poles and zero of system closed-loop transfer function.

To calculate, for example, the complex root sensitivity matrix $S_{1,a}$ corresponding to the small variation of system parameter $a = 1.45$, the following terms of Eqs. 13 and Eqs. 14 should be calculated using Eqs. 2, values $\zeta_1 = 0.5$ and $\omega_1 = 0.72$, table of the Chebyshev functions, and numerical values of system parameters

$$A_{11} = 0.23, \quad A_{12} = 0.42, \quad A_{21} = -0.28, \quad A_{22} = 0.47, \quad Q_1^a{}_{,1} = -0.135, \quad Q_{1,2}^a = 0$$

Now, finding the inverse of A_i and then using Eq. 12, one obtains

(20a) $$S_{1,a} = (0.28, 0.18)$$

To calculate the sensitivity matrices $S_{1,b}$, $S_{1,c}$, and $S_{1,K}$, the procedure is to be repeated bearing in mind that the inverse of A_i stays unchanged regarding each parameter. In doing so, one obtains

(20b) $$S_{1,b} = (1.35, -0.19), \quad S_{1,c} = (1.08, \text{ i } 0.41), \quad S_{1,K} = (-0.43, 0.36)$$

The correspondiig real root ($\sigma_1 = -0.18$) sensitivities may be calculated applying Eqs. 2 and Eq. 16, to obtain

(21) $$S^{\sigma}_{1,a} = -0.65, \quad S^{\sigma}_{1,b} = 0.15, \quad S^{\sigma}_{1,c} = 0.59, \quad S^{\sigma}_{1,K} = 0.87$$

In like manner, the real zero sensitivities are:

(22) $$S^{z}_{1,a} = 1.45, \quad S^{z}_{1,b} = S^{z}_{1,c} = S^{z}_{1,K} = 0$$

Comparing the obtained results in Eqs. 20a, 20b, 21, and 22, one can conclude how the small changes in system parameters effect the corresponding poles and zero and therefore these results may advantageously be used in system synthesis and adaptation. For example, according to the definition of both complex root sensitivity matrix and real root sensitivity (Eqs. 4 and 5, respectively), the calculated sensitivities indicate that a decrease in the values of parameters a, b, and c or an increase in the gain factor K will lead to a lower system stability and so on.

CONCLUSION

The proposed sensitivity analysis of linear continuous systems was used in the study of system performance characteristics due to small system parameter perturbations. Applying the suitable defined sensitivities, the effects of small parameter variations on the poles and zeros of system transfer function has been considered and therefore the presented sensitivity analysis may appear useful whenever it is possible to describe a system with a location of poles and zeros of a corresponding system transfer function. A significant advantage of the proposed sensitivity analysis lies in the fact that the Chebyshev functions are introduced. Thus, the analytical procedure, which may be performed in the real domain, is extremely simplified and becomes convenient for the straightforward application of both analog and digital computers.

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