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A CORRECT CONSIDERATION OF THE CLASSICAL THEORY OF NORMAL ZEEMAN EFFECT*

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The normal Zeeman effect can be treated by classical and quantum theory. While in quantum treatmant the results are obtained directly from Schrödinger's equation [1], or from Dirac's relativistic equation [2], classical theory obtained the same results, either by analysing the motion of electrons in the central electric field of a nucleus and in homogeneous magnetic field [3], [4], [5], or by using Lorentz's model of linear harmonic oscillator [6].

Exact analysis of the electron motion in the central electric field and nonstationary homogeneous magnetic field is very complicated. In the literature the simplified analysis of motion can be find, where the electron is supposed to have a circular orbit in electric field of a nucleus when there is no magnetic field. Then noncorrectly and nonreliably** one takes that the radius of the circle does not change with the rising of magnetic field which is normal to the plane of motion, but the angular velocity does. However, the radius of the circular orbit of electron is changed because of existance of magnetic field.

In this paper it is shown that the radius of the circular orbit is changed when the magnetic field rises slowly, because, in this case, the electron orbit is supposed to become approximately a circle. The analytical expression for the ratio r_1/r_0 can be found as the function of the magnetic field. r_0 is the radius of circular orbit in electric field, and r_1 is the radius of circular orbit in electric and magnetic fields. The expression for the change of frequency is obtained directly from the principle of conservation of angular momentum p_θ. For the cases of practical interest it is found that $r_1/r_0 \approx 1$.

A SIMPLIFIED ANALYSIS OF THE ELECTRON MOTION IN THE CENTRAL ELECTRIC AND NONSTATIONARY HOMOGENEOUS MAGNETIC FIELD

The equation of motion of a charged particle with charge q and mass m in the nonrelativistic case is:

$$(1) \qquad m\frac{d^2\vec{r}}{dt^2} = q\left(\vec{\mathcal{E}} + \frac{1}{c}\frac{d\vec{r}}{dt}\times\vec{B}\right),$$

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** In the reference [3] there is the statement that the radius of circular orbit of electron does not change when we switch on the magnetic field. This statement is not correct.

where $\vec{r}$ is the radius of the electron position, and c the velocity of light in vacuo.

Let us consider the motion of an electron in the central electric field of a nucleus with the charge $+Ze$ which we take at rest, and in homogeneous magnetic field $B_{(t)}$ which has the same direction as $+z$ axis. (Fig. 1.a; 1.b).

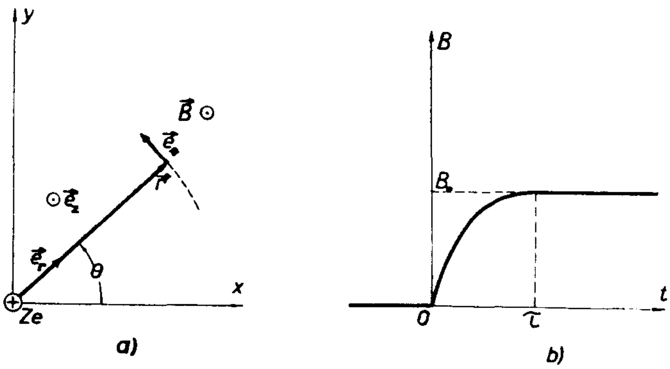

Fig.1.

The electric field in this case is

(2) $$\vec{\mathfrak{E}} = -\operatorname{grad}\varphi - \frac{1}{c}\frac{\partial \vec{\mathcal{A}}}{\partial t},$$

where φ and $\vec{\mathcal{A}}$ are electric scalar and magnetic vector potentials respectively, or

(3) $$\vec{\mathfrak{E}} = \frac{Ze}{r^2}\cdot\vec{e}_r - \frac{r}{2c}\frac{\mathrm{d}B}{\mathrm{d}t}\cdot\vec{e}_\theta .$$

Substituting this value in the Eq. (1) we obtain

(4) $$m\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = -\frac{Ze^2}{r^2}\vec{e}_r + \frac{e\,r}{2c}\frac{\mathrm{d}B}{\mathrm{d}t}\cdot\vec{e}_\theta - \frac{e}{c}\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\times\vec{e}_z\right)\cdot B.$$

As it is known

(5) $$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}t}\cdot\vec{e}_r + r\frac{\mathrm{d}\theta}{\mathrm{d}t}\cdot\vec{e}_\theta$$

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \left[\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right]\vec{e}_r + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\vec{e}_\theta ,$$

then the Eq. (4) written in θ direction has the form

$$\frac{m}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)=\frac{e\,r}{2c}\frac{\mathrm{d}B}{\mathrm{d}t}+\frac{e}{c}\frac{\mathrm{d}r}{\mathrm{d}t}\cdot B \tag{6}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\frac{\mathrm{d}\theta}{\mathrm{d}t}-\frac{1}{2}\frac{eB}{c}r^2\right)=0. \tag{7}$$

The angular momentum becomes

$$mr^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}-\frac{1}{2}\frac{eB}{mc}\right)=mr^2\left(\omega-\frac{1}{2}\omega_c\right)=\text{Const.}, \tag{8}$$

and it is the integral of the motion, and indepedent of the manner of magnetic field rising from $B=0$ to $B=B_0$, where $\omega_c=\dfrac{eB}{mc}$ is cyclotron angular velocity.

If the magnetic field has the same direction as $-z$ axis, the angular momentum, considering the Eq. (6), has the form

$$mr^2\left(\omega+\frac{1}{2}\omega_c\right)=\text{Const.} \tag{9}$$

The component of the Eq. (1) in radial direction is inhomogeneous differential equation of the second order, and the analysis of an electron motion in this configuration of fields is very complicated. That is the reason why we observe only the case in which the electron has circular orbit of radius r_0 and angular velocity ω_0 for $t<0$, and circular orbit of radius r_1 and angular velocity ω_1 for $t>\tau$. In this case it is necessary that the magnetic field rises slowly i. e. that $\dfrac{2\pi}{\omega_0}\ll\tau$, where τ is the time for which the magnetic field comes to constant value.

For $t<0$ we can write

$$\frac{Ze^2}{r_0^2}=mr_0\,\omega_0^2, \tag{10}$$

and for $t>\tau$, in the case when $v_0=r_0\,\omega_0$ is in the direction of $\vec{e_\theta}$ we obtained

$$\frac{Ze^2}{r_1^2}+\frac{e}{c}r_1\,\omega_1\,B_0=mr_1\,\omega_1^2. \tag{11}$$

As the angular momentum in θ direction is an integral of motion, we can write

$$mr_0^2\,\omega_0=mr_1^2\left(\omega_1-\frac{1}{2}\omega_{c0}\right), \tag{12}$$

where $\omega_{c0}=\dfrac{eB_0}{mc}$.

From the Eqs. (11) and (12) we can eliminate ω_1 and, by using the Eq. (10), we can write the following equation for r_1:

$$r_1^4 + 4\left(\frac{\omega_0}{\omega_{c0}}\right)^2 \cdot r_0^3 \cdot r_1 - 4\left(\frac{\omega_0}{\omega_{c0}}\right)^2 \cdot r_0^4 = 0. \tag{13}$$

The same result is obtained in the case when we follow the motion of an electron in the magnetic field whose direction is the same as the direction of $-z$ axis and the velosity v_0 has the same direction as in previous case, or vice versa.

Then we have

$$m r_0^2\, \omega_0 = m r_1^2 \left(\omega_1 + \frac{1}{2}\,\omega_c\right) \tag{14}$$

and

$$\frac{Ze^2}{r_1^2} - \frac{e}{c}\, r_1\, \omega_1\, B_0 = m r_1\, \omega_1^2. \tag{15}$$

From the Eqs. (14) and (15), by using the Eq. (10), the same result, as in the Eq. (13), can be obtained.

The Eq. (13) has one positive, one negative and two complex roots. Solving this equation we find for r_1

$$r_1 = r_0\left(\frac{\omega_0}{\omega_{c0}}\right)^{2/3} \cdot \left[\left(F^{-\frac{1}{2}} - F\right)^{\frac{1}{2}} - F^{\frac{1}{2}}\right], \tag{16}$$

where

$$F = \frac{1}{2}\left\{\left[1 + \left(1 + \frac{64}{27}\,\frac{\omega_{c0}^2}{\omega_0^2}\right)^{1/2}\right]^{\frac{1}{3}} + \left[1 - \left(1 + \frac{64}{27}\,\frac{\omega_{c0}^2}{\omega_0^2}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}\right\}. \tag{17}$$

It can be shown that $\lim\limits_{\omega_{c0}\to\infty} r_1 = 0$ and $\lim\limits_{\omega_{c0}\to\infty} r_1 = r_0$.

The dependence of the ratio r_1/r_0 in function of $\dfrac{\omega_0}{\omega_{c0}}$ has been shown on the fig. 2, as follows from the Eq. (16).

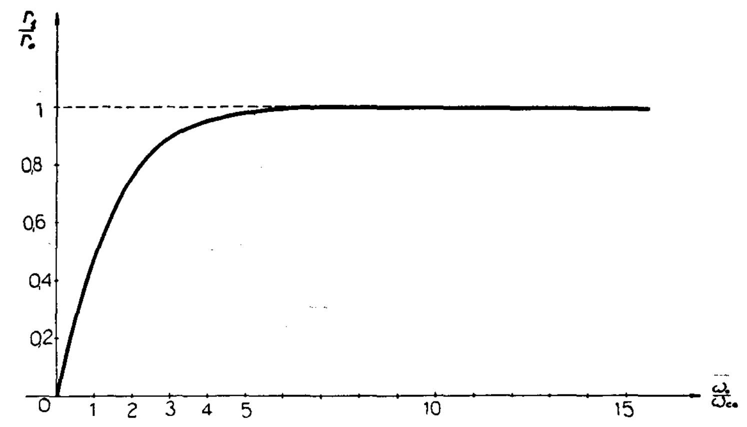

Fig. 2.

In the case of the first ten orbits in the Hydrogen atoms with magnetic field $B=1\ T$, the ratio $\frac{\omega_0}{\omega_{c0}}$ has the values from 10^2 to 10^5, but for the atoms similar to the Hydrogen atom and lower magnetic fields, this ratio is bigger. Then from the Eq. (16) it could be seen that $r_1 \approx r_0$. (It comes from the Eq. (16) that for $\frac{\omega_0}{\omega_{c0}}=10$ the ratio $r_1/r_0=0{,}99$.)

When $r_1=r_0$, from the Eqs. (12) and (14), we obtained

$$\omega_1=\omega_0 \pm \frac{1}{2}\omega_{c0}, \quad \text{i. e.,} \quad \nu_1=\nu_0 \pm \frac{e B_0}{4\pi mc}$$

and these are the same results for normal Zeeman effect as in the quantum theory.

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