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GENERAL SOLUTION OF A FUNCTIONAL EQUATION*

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We consider the functions defined on an arbitrary set S, which take their values in a certain abelian group M. The independent variables will be denoted by x_i, $i=1, \ldots, n$.

Let C_k^n be the set of all strictly increasing mappings of the set $\{1, 2, \ldots, k\}$ into $\{1, 2, \ldots, n\}$. We shall solve the functional equation

$$\sum_{c \in C_k^n} f_c(x_{c(1)}, \ldots, x_{c(k)}) = 0 \qquad (n > k > 0) \tag{1}$$

where all functions f_c are unknown.

Theorem. *The general solution of the functional equation* (1) *is given by*

$$f_c(x_{c(1)}, \ldots, x_{c(k)}) = \sum_{s \in C_{k-1}^k} F_c^s(x_{cs(1)}, \ldots, x_{cs(k-1)}) \qquad (c \in C_k^n) \tag{2}$$

where $cs(i) = c(s(i))$. *The functions* F_c^s *are arbitrary but subjected to the following conditions*

$$\sum_{cs=t} F_c^s(x_{t(1)}, \ldots, x_{t(k-1)}) = 0 \qquad (t \in C_{k-1}^n) \tag{3}$$

where the sum is extended over all c *and* s *such that* $cs = t$.

Proof. Let f_c be defined by (2) and (3). Then we obtain

$$\begin{aligned}
&\sum_{c \in C_k^n} f_c(x_{c(1)}, \ldots, x_{c(k-1)}) \\
&\quad = \sum_{c \in C_k^n} \sum_{s \in C_{k-1}^k} F_c^s(x_{cs(1)}, \ldots, x_{cs(k-1)}) \\
&\quad = \sum_{t \in C_{k-1}^n} \sum_{cs=t} F_c^s(x_{t(1)}, \ldots, x_{t(k-1)}) \\
&\quad = 0.
\end{aligned}$$

Hence, such functions satisfy the functional equation (1).

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Conversely, if $f_c\,(c\in C_k^n)$ is any solution of (1) we have to prove that the functions f_c admit the representation (2) with the conditions (3). Let us put in (1) $x_i=\text{const}$ if $i\neq c(j)$ $(j=1,\ldots,k)$ where c is fixed. Then (1) gives

$$f_c(x_{c(1)},\ldots,x_{c(k)})=\sum_{s\in C_{k-1}^k} G_c^s(x_{cs(1)},\ldots,x_{cs(k-1)}). \tag{4}$$

For any $t\in C_{k-1}^n$ let

$$H_t(x_{t(1)},\ldots,x_{t(k-1)})=\sum_{cs=t} G_c^s(x_{t(1)},\ldots,x_{t(k-1)}). \tag{5}$$

The equation (1) can be written in the form

$$\sum_{t\in C_{k-1}^n} H_t(x_{t(1)},\ldots,x_{t(k-1)})=0. \tag{6}$$

Let T' be the set of all $t\in C_{k-1}^n$ such that $H_t\not\equiv 0$ and $T''=C_{k-1}^n\setminus T'$. The equation (6) reduces to

$$\sum_{t\in T'} H_t(x_{t(1)},\ldots,x_{t(k-1)})=0. \tag{7}$$

We can suppose that the number r of elements of the set T' is taken to be minimal over all representations (4) of the functions f_c. If $r=0$ we can put $G_c^s=F_c^s$. The case $r=1$ is impossible since (7) holds. Therefore we can suppose that $r>1$. Let t be some fixed element of T'. Putting in (7) $x_i=\text{const}$ if $i\neq t(j)$ $(j=1,\ldots,k-1)$, we get

$$H_t(x_{t(1)},\ldots,x_{t(k-1)})=-\sum J_c^s(x_{t(i_1)},\ldots,x_{t(i_m)}) \tag{8}$$

where $J_c^s(x_{t(i_1)},\ldots,x_{t(i_m)})$ $(m\leqslant k-2)$ is obtained from $G_c^s(x_{cs(1)},\ldots,x_{cs(k-1)})$ bi putting $x_i=\text{const}$ for all i but $t(1),\ldots,t(k-1)$. The sum on the right-hand side of (8) is extended over certain (not all) pairs of indices c and s.

Consider a certain summand $J_{c_0}^{s_0}$ on the right-hand side of (8). Let $u_0\in C_k^n$ and $v_0\in C_{k-1}^k$ be such that $u_0v_0=t$. We can form a sequence of ordered pairs

$$(u_0,v_0),(u_0,w_0),(u_1,v_1),(u_1,w_1),\ldots,(u_p,w_p)$$

which satisfy the following conditions

1° $u_i\in C_k^n,\ v_i\in C_{k-1}^k,\ w_i\in C_{k-1}^k$;

2° $(u_p,w_p)=(c_0,s_0)$;

3° $u_{i-1}w_{i-1}=u_iv_i$ $(i=1,\ldots,p)$;

4° the sequence $u_iw_i(1),\ldots,u_iw_i(k-1)$ $(i=0,1,\ldots,p)$ contains the sequence $t(i_1),\ldots,t(i_m)$ as a subsequence.

Let us put

$$\overline{G}_{u_i}^{v_i}=G_{u_i}^{v_i}+J_{c_0}^{s_0},\quad \overline{G}_{u_i}^{w_i}=G_{u_i}^{w_i}-J_{c_0}^{s_0}\qquad (i=0,1,\ldots,p). \tag{9}$$

We remark that (4) remains valid if we substitute $\overline{G}_{u_i}^{v_i}$ and $\overline{G}_{u_i}^{w_i}$ instead of $G_{u_i}^{v_i}$ and $G_{u_i}^{w_i}$, respectively. Also, if $w \in T''$ i. e. $H_w \equiv 0$ then also $\overline{H}_w \equiv 0$. Further, we have $\overline{H}_t = H_t + J_{c_0}^{s_0}$.

If the same procedure is applied to all summands of the right-hand member of (8), we conclude that the new function H_t is identically zero. This contradicts the minimum property of r. Hence, $r = 0$ which proves the theorem.

Example. If $n = 4$ and $k = 3$ the equation (1) is

$$f(x_1, x_2, x_3) + g(x_1, x_2, x_4) + h(x_1, x_3, x_4) + i(x_2, x_3, x_4) = 0.$$

Its general solution is given by

$$f(x_1, x_2, x_3) = f_1(x_1, x_2) + f_2(x_1, x_3) + f_3(x_2, x_3),$$
$$g(x_1, x_2, x_4) = g_1(x_1, x_2) + g_2(x_1, x_4) + g_3(x_2, x_4),$$
$$h(x_1, x_3, x_4) = h_1(x_1, x_3) + h_2(x_1, x_4) + h_3(x_3, x_4),$$
$$i(x_2, x_3, x_4) = i_1(x_2, x_3) + i_2(x_2, x_4) + i_3(x_3, x_4),$$

where

$$f_1(x_1, x_2) + g_1(x_1, x_2) = 0, \quad f_2(x_1, x_3) + h_1(x_1, x_3) = 0,$$
$$f_3(x_2, x_3) + i_1(x_2, x_3) = 0, \quad g_2(x_1, x_4) + h_2(x_1, x_4) = 0,$$
$$g_3(x_2, x_4) + i_2(x_2, x_4) = 0, \quad h_3(x_3, x_4) + i_3(x_3, x_4) = 0.$$

Hence, we can take $f_1, f_2, f_3, g_2, g_3, h_3$ to be arbitrary and $g_1 = -f_1$, $h_1 = -f_2$, $i_1 = -f_3$, $h_2 = -g_2$, $i_2 = -g_3$, $i_3 = -h_3$.

Remark. This problem was raised recently by P. M. Vasić in the Institute of Mathematics of Belgrade. He has solved it for some special values of k. His solution is not given in the symmetric form.