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NOTE ON A q-IDENTITY

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Making use of the Fourier series of the Jacobi elliptic functions *Fempl* [2] has obtained the identity

$$\sum_{r=1}^{\infty}\left(\frac{\sqrt{q^{2r-1}}}{1-q^{2r-1}}\right)^2+\sum_{r=1}^{\infty}\left(\frac{\sqrt{q^{2r-1}}}{1+q^{2r-1}}\right)^2=2\left(\sum_{r=1}^{\infty}\frac{\sqrt{q^{2r-1}}}{1+q^{2r-1}}\right)^2. \tag{1}$$

Since

$$\sum_{r=1}^{\infty}\frac{q^{2r-1}}{(1-q^{2r-1})^2}=\sum_{r=1}^{\infty}\sum_{s=0}^{\infty}(s+1)\,q^{(2r-1)(s+1)},$$

$$\sum_{r=1}^{\infty}\frac{q^{2r-1}}{(1+q^{2r-1})^2}=\sum_{r=1}^{\infty}\sum_{s=0}^{\infty}(-1)^s(s+1)\,q^{(2r-1)(s+1)},$$

it follows that the left member of (1) is equal to

$$\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(1-(-1)^s)\,s\,q^{(2r+1)s}$$

$$=2\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(2s-1)\,q^{(2r-1)(2s-1)}$$

$$=2\sum_{n=1}^{\infty}\sigma(2n-1)\,q^{2n-1},$$

where

$$\sigma(n)=\sum_{d\mid n}d.$$

On the other hand, since

$$\sum_{r=1}^{\infty}\frac{\sqrt{q^{2r-1}}}{1+q^{2r-1}}=\sum_{r=1}^{\infty}\sum_{s=0}^{\infty}(-1)^s\,q^{\frac{1}{2}(2r-1)(2s+1)}$$

$$=\sum_{n=1}^{\infty}\rho(2n-1)\,q^{\frac{1}{2}(2n-1)},$$

where

$$\rho(2n-1)=\sum_{d\mid 2n-1}(-1)^{\frac{1}{2}(d-1)}$$

it is clear that the right member of (1) is equal to

$$\sum_{n=1}^{\infty} q^{2n-1} \sum_{\substack{r+s=2n \\ r\geqslant 1,\, s\geqslant 1}} \rho(2r-1)\,\rho(2s-1).$$

Thus (1) is equivalent to

$$\sum_{\substack{r+s=2n \\ r\geqslant 1,\, s\geqslant 1}} \rho(2r-1)\,\rho(2s-1)=\sigma(n). \tag{2}$$

For a direct proof of (2) see *Bachmann* [1, p. 349] or *Landau* [3, p. 110].

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