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ON KINEMATIC INTERPRETATION OF AN IMPORTANT DIFFERENTIAL EQUATION

by

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We shall give here an illustration of the differential equation $\left(\frac{d\rho}{d\theta}\right)^2 + \rho^2 = f(\theta)$, which could serve as a starting idea how to form a mecanical or electrical device for the construction of integral curves for this equation.

1. Problem. *The point P is moving on a plane curve $R = R(\theta)$ at the velocity $u = u(t)$. Following it, in the same plane, from a certain position and at a definite moment, the point M begins to move at the velocity $v = v(t)$, so that its radius vector is directed constantly towards the point P (Fig. 1). Determine the trajectory of the point M.*

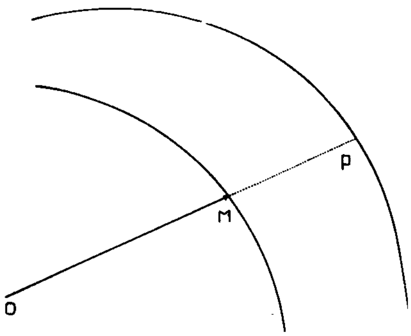

Fig. 1.

Solution. Starting with the vector equation:

$$d\vec{A} = \vec{A}_\circ \, da + a \cdot d\vec{A}_\circ \tag{1}$$

where $\vec{A} = a \cdot \vec{A}_\circ$ is the vector radius of a point, $\vec{A}_\circ$ the unit vector and a the magnitude of the vector $\vec{A}$, we get multiplying scalarly:

$$(d\vec{A})^2 = (da)^2 + a^2 (d\vec{A}_\circ)^2. \tag{2}$$

Out of this follows:

(3) $$(dl)^2 = (da)^2 + a^2 (d\theta)^2$$

$d\theta$ is the element of the polar angle and dl the element of the arc described by the end of the radius vector.

If we differentiate (3) with t, we get:
for the point P:

(4) $$u^2 = \left[\left(\frac{dR}{d\theta}\right)^2 + R^2\right]\left(\frac{d\theta}{dt}\right)^2$$

and for the point M:

(5) $$v^2 = \left[\left(\frac{d\rho}{d\theta}\right)^2 + \rho^2\right]\left(\frac{d\theta}{dt}\right)^2.$$

The problem is reduced then, with the elimination of the parametar t from (4) and (5) to the integration of the equation:

(6) $$\left(\frac{d\rho}{d\theta}\right)^2 + \rho^2 - \frac{v^2}{u^2}\left[\left(\frac{dR}{d\theta}\right)^2 + R^2\right] = 0 \quad \text{or} \quad \left(\frac{d\rho}{d\theta}\right)^2 + \rho^2 = f(\theta).$$

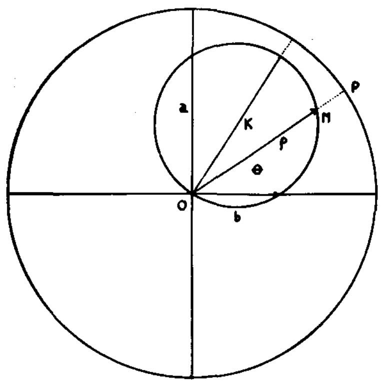

Fig. 2

2. A particular case. If $R = C$ (const.), $v = C_1$ (const.), $u = C_2$ (const.), the equations (4) and (6) read respectively:

$$C\theta = u \cdot t, \quad \left(\frac{d\rho}{d\theta}\right)^2 + \rho^2 - \frac{v^2 C^2}{u^2} = 0 \quad \text{or} \quad \frac{d\rho}{d\theta} = \pm\sqrt{k^2 - \rho^2} \quad \left(k = \frac{Cv}{u}\right)$$

The tangent being oriented, we can take only the sign + and integrating we obtain:

$$\rho = k \sin(\theta + \alpha)$$

$\alpha = \text{arc} \sin \frac{b}{k}$, b is the radius vector of the point M for the moment $t = 0$; on the picture the case is with $b < a$ (Fig. 2).

That is the circle with the diameter k and its center being at the point $r = \frac{k}{2}, \theta = \frac{\pi}{2} - \alpha$.

When $k<C$, $v<u$ the point M will never reach the point P.

If $k=C$, $v=u$ the points M and P will meet each other at the point $r=C, \theta=\frac{\pi}{2}-\text{arc sin}\frac{b}{C}$ after a space of time $t=\frac{C\left(\frac{\pi}{2}-\text{arc sin}\frac{b}{C}\right)}{u}$

In the case $k>C$, $v>u$, the point M will overtake the point P after a space of time $t=\frac{C\left(\text{arc sin}\frac{u}{v}-\alpha\right)}{u}$ at the point $r=C$, $\theta=\text{arc sin}\frac{u}{v}-\alpha$.

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[1] For equation: $\left(\frac{d\rho}{d\theta}\right)^2+\rho^2=f(\theta)$ cf.:

D. S. Mitrinović: *Bulletin de l'Académie des sciences serbe*, t. 1 (1933), p. 107–117; t. 2 (1935), p. 61–65; t. 3 (1936), p. 7—19; t. 6 (1939), p. 99–120.

D. S. Mitrinović: *Comptes Rendus de l'Académie des sciences de Paris*, t. 204 (1937), p. 1706–1708; t. 205 (1937), p. 1194—1196.

REZIME

O KINEMATIČKOJ INTERPRETACIJI JEDNE VAŽNE DIFERENCIJALNE JEDNAČINE

Slobodan V. Pavlović

Ovde smo izložili jednu kinematičku ilustraciju diferencijalne jednačine: $\left(\frac{d\rho}{d\theta}\right)^2+\rho^2=f(\theta)$, na osnovu koje bi se mogao konstruisati mehanizam za konstrukciju integralnih krivih gornje jednačine.

Problem: Tačka P se kreće po ravnoj krivoj $R=R(\theta)$ brzinom $u=u(t)$. Za njom, u istoj ravni iz izvesnog položaja i u nekom određenom trenutku, počne da se kreće tačka M datom brzinom $v=v(t)$. Radijus vektor tačke M je stalno naperen prema tački P. Odrediti trajektoriju tačke M.

Rešenje: Polazeći od vektorske jednačine (1) gde je $\vec{A}=a\cdot\vec{A}_0$ radijus vektor neke tačke, $\vec{A}_0$ ort a a intenzitet vektora $\vec{A}$, dižući na kvadrat skalarno levu i desnu stranu, dolazi se do jednačina (2) i zatim (3) ($d\theta$ je element polarnog ugla a dl element luka koji opisuje kraj radijusa vektora $\vec{A}$).

Ako diferenciramo jednačinu (3) po t dobijamo jednačine (4) za tačku P i (5) za tačku M.

Eliminacijom parametra t dolazi se do diferencijalne jednačine (6).

Na kraju je posmatran specijalan slučaj kad se tačka P kreće po krugu i kad su brzine obeju tačaka konstantne, kao i uslovi pod kojima se tačke P i M mogu susresti.