PART-PRODUCTS OF *S*-RESTRICTED INTEGER COMPOSITIONS

Eric Schmutz, Caroline Shapcott

If S is a cofinite set of positive integers, an "S-restricted composition of n" is a sequence of elements of S, denoted $\vec{\lambda} = (\lambda_1, \lambda_2, \dots)$, whose sum is n. For uniform random S-restricted compositions, the random variable $\mathbf{B}(\vec{\lambda}) = \prod_i \lambda_i$ is asymptotically lognormal. (A precise statement of the theorem includes an error term to bound the rate of convergence.) The proof is based upon a combinatorial technique for decomposing a composition into a sequence of smaller compositions.

1. INTRODUCTION

A composition of n is a sequence of positive integers whose sum is n. Hitczenko made the following observation: if $\Gamma_1, \Gamma_2, \dots$ are independent random variables with Geometric(1/2) distributions, and if $\tau = \min \left\{ t : \sum_{i=1}^{t} \Gamma_i \geq n \right\}$, then

$$\left(\Gamma_1, \Gamma_2, \dots, \Gamma_{\tau-1}, n - \sum_{i=1}^{\tau-1} \Gamma_i \right)$$

is a uniform random composition of n. Using this fact, HITCZENKO and others were able to determine the asymptotic distributions of a variety of random variables defined on the space of compositions of n with a uniform probability measure [**1, 16, 17, 18, 19, 20, 21**].

In her thesis [**32**], SHAPCOTT considered the random variable $\mathbf{B}(\vec{\lambda}) = \prod_i \lambda_i$, the product of the parts of $\vec{\lambda}$. Because of Hitczenko's observation, it was straightforward for her to use known results on random index summation [**13, 30**] to prove

that **B** is asymptotically lognormal. The goal of this paper is to extend Shapcott's results to a more general setting where Hitczenko's observation is *not* applicable.

In recent years, there has been a resurgence of interest in compositions with restrictions on the part sizes. The easiest case is "S-restricted compositions," i.e. compositions whose parts are all elements of a fixed subset $S \subset \mathbb{Z}_+$. Hence there have been papers on compositions with no parts of size 2 [**8**], compositions with no parts of size k [**7**], compositions with parts from the interval $(1, k)$ [**6**], compositions with parts greater than or equal to d [**5**], compositions with parts equal to either a or b [**5**], compositions with parts from an arbitrary finite set [**27**], and compositions with parts from an arbitrary (not necessarily finite) set [**2, 14**]. More complicated restrictions have also been considered, such as restrictions on the differences between successive parts [**3, 25**] and restrictions on the parts' multiplicities [**24, 26**]. Interested readers are referred to the recent book by HEUBACH and MANSOUR [**15**] for more background.

In [**31**], SHAPCOTT studied the asymptotic distribution of **B** for uniform random 1-free compositions of n, i.e. the case $S = \{k \in \mathbb{Z} : k \geq 2\}$. In this case, it does not seem possible to generate random compositions using a stopped sequence of independent random variables. It is straightforward to replace the geometric variables with 1-omitting analogues, but there is no obvious way around the fact that the putative last part $n - \sum_{i=1}^{\tau-1} \Gamma_i$ need not lie in S. Furthermore, the randomly-generated compositions are not all equally likely to be chosen. Shapcott was able to circumvent these difficulties by embedding the set of 1-free compositions in a more tractable auxiliary space and doing the hard calculations there. The method of proof was completely different from the methods in this paper.

This paper concerns S-restricted compositions of n in the case where $S \subset \mathbb{Z}_+$ is any proper cofinite set of positive integers. We prove that, for random S-restricted compositions of n, **B** is asymptotically lognormal. As in [**22**], we deduce bounds on the rate of convergence using methods that are ultimately based on the Berry-Essen inequality. However, our proof involves a blocking argument, similar to that of BERNSTEIN [**4**] and MARKOV (see page 376 of [**9**]), for decomposing a composition into a sequence of smaller compositions.

Theorem 1. *Let P_n denote the uniform probability measure on the set of S-restricted compositions of n. Then there exist positive constants a_1, b_1, c; constants a_0, b_0; and sequences of constants $\langle \mu_n \rangle_{n=1}^{\infty}, \langle \sigma_n \rangle_{n=1}^{\infty}$; such that the following three statements hold:*

- $\mu_n = a_1 n + a_0 + o(1)$,

- $\sigma_n^2 = b_1 n + b_0 + o(1)$, *and*

- *For all n,*

$$\sup_x \left| P_n \left(\frac{\log \mathbf{B} - \mu_n}{\sigma_n} \leq x \right) - \Phi(x) \right| \leq \frac{c \, (\log n)^{1/3}}{n^{1/6}}.$$

Throughout this manuscript, we denote the set of S-restricted compositions of n as Λ_n and an individual composition of n as $\vec{\lambda}$. P_n denotes the uniform probability measure on Λ_n, and E_n denotes the expected value with respect to P_n. If F is a formal power series in x, then we write $[x^n]F$ to denote the coefficient of x^n in F.

2. NUMBER OF COMPOSITIONS

At several points in the proof we need estimates for the cardinality of the sample space Λ_n. This kind of calculation can be considered folklore since it is clearly known to experts, but it is hard to know who to credit (see page 297 of [11], for example, and the rather general results in [3]). We present an asymptotic formula that will serve the needs of this paper.

We begin by defining S to be an arbitrary proper cofinite set of positive integers and M to be the largest element of $\bar{S} = \mathbb{Z}_+ - S$. Define

$$F(x,t) = 1 - \sum_{k \in S} k^t x^k \quad \text{and} \quad f(x) = F(x,0) = 1 - \sum_{k \in S} x^k.$$

Lemma 1. *The smallest magnitude root of $f(x) = 1 - \displaystyle\sum_{k \in S} x^k$ is real, lies in the interval $\left(\dfrac{1}{2}, 1\right)$, and has multiplicity one.*

Proof. First we verify that f has a real root p in the interval $\left(\dfrac{1}{2}, 1\right)$. The function f is continuous and strictly decreasing on $(0, 1)$. Note that $f\left(\dfrac{1}{2}\right)$ is strictly positive and that $\displaystyle\lim_{x \to 1^-} f(x) = -\infty$. Therefore there is a unique positive real root p in the interval $\left(\dfrac{1}{2}, 1\right)$.

Next we use Rouché's theorem to verify that f has no other roots in $|x| < p$. Let g be the constant function $g(x) = 1$, which obviously has no zeros in $|x| < p$. Let ϵ be an arbitrarily small positive number, and observe that $\displaystyle\sum_{k \in S} p^k = 1$. Then, for $|x| = p - \epsilon$,

$$|f(x) - g(x)| = \left| -\sum_{k \in S} x^k \right| \le \sum_{k \in S} |x|^k < \sum_{k \in S} p^k = 1 = |g(x)|.$$

By Rouché's theorem, f has no roots inside the circle $|x| = p - \epsilon$.

We use proof by contradiction to verify that no other root of $f(x)$ has magnitude *equal* to p. Suppose $\hat{p}$ is a root of $f(x)$ such that $|\hat{p}| = p$ and $\hat{p} \ne p$. Because $\overline{S}$ is finite, S includes odd elements and

$$f(-p) = 1 - \sum_{k \in S} (-p)^k > 1 - \sum_{k \in S} p^k = 0.$$

Therefore $\hat{p}$ is not real and we have $\hat{p} = p(\cos\theta + i\sin\theta)$ with $0 < |\theta| < 2\pi$. Since $f(\hat{p}) = 0$, the real part of $f(\hat{p})$ is also zero:

$$0 = \mathrm{Re}\left(1 - \sum_{k\in S}\hat{p}^k\right) = 1 - \sum_{k\in S}p^k\cos(k\theta)$$

$$= \left(1 - \sum_{k\in S}p^k\right) + \sum_{k\in S}p^k\big(1 - \cos(k\theta)\big) = 0 + \sum_{k\in S}p^k\big(1 - \cos(k\theta)\big).$$

Because $1 - \cos(k\theta) \geq 0$, each term in the sum must be zero. Therefore, for all $k \in S$, there is an $\ell_k \in \mathbb{Z}$ such that $k\theta = 2\pi\ell_k$. Because $\bar{S}$ is finite, we can choose k_0 such that k_0 and $k_0 + 1$ are both elements of S, but then

$$\theta = (k_0 + 1 - k_0)\theta = 2\pi(\ell_{k_0+1} - \ell_{k_0}).$$

Since $\ell_{k_0+1} - \ell_{k_0} \in \mathbb{Z}$, this contradicts the fact that $0 < |\theta| < 2\pi$.   □

The following lemma is proven in [**31**] and is needed for Theorems 2, 3, and 4.

Lemma 2. *The moments of* **B** *are given by*

$$E_n(\mathbf{B}^t) = \frac{1}{|\Lambda_n|}[x^n]\frac{1}{1 - \displaystyle\sum_{k\in S}k^t x^k}.$$

Theorem 2. *Let p be the smallest root of $f(x)$ and let r be the magnitude of the second smallest root. Then*

$$|\Lambda_n| = \frac{1}{p^n\displaystyle\sum_{k\in S}kp^k} + O\left(\frac{n^{M-1}}{r^n}\right).$$

Proof. By Lemma 2 with $t = 0$,

$$|\Lambda_n| = [x^n]\frac{1}{1 - \displaystyle\sum_{k\in S}x^k} = [x^n]\frac{1}{f(x)}.$$

Observe that

$$f(x) = 1 - \left(\sum_{k\in\mathbb{Z}_+}x^k - \sum_{k\in\bar{S}}x^k\right) = 1 - \frac{x}{1-x} + P_M(x)$$

where M is the largest element of $\bar{S}$ and $P_M(x)$ signifies a polynomial of degree M. Multiplying both sides by $1 - x$, we have

$$(1-x)f(x) = 1 - 2x + P_{M+1}(x).$$

Therefore $(1-x)f(x)$ has exactly $M+1$ roots (one of which is $x = p$) and $\dfrac{1}{f(x)} = \dfrac{1-x}{(1-x)f(x)}$ is rational. We can therefore apply standard methods for approximating the coefficients of rational generating functions.

Define r_i for $i = 1, \ldots, M$ to be the remaining roots of f, and set k_i equal to the multiplicity of r_i. Without loss of generality, assume $|r_i| \leq |r_{i+1}|$. Then we use Lemma 1 to write

$$|\Lambda_n| = [x^n]\frac{1-x}{(1-x)f(x)}$$

$$= [x^n]\frac{C_0}{x-p} + [x^n]\sum_i \left(\frac{C_{i,1}}{x-r_i} + \frac{C_{i,2}}{(x-r_i)^2} + \cdots + \frac{C_{i,k_i}}{(x-r_i)^{k_i}}\right)$$

where

$$C_0 = \lim_{x \to p} \frac{x-p}{1 - \sum_{k \in S} x^k} = \lim_{x \to p} \frac{1}{-\sum_{k \in S} kx^{k-1}} = \frac{1}{-\sum_{k \in S} kp^{k-1}}.$$

We can use the fact that

$$(1) \qquad [x^n]\frac{C_{i,j}}{(x-r_i)^j} = [x^n]\frac{C_{i,j}}{(-r_i)^j}\left(1 - \frac{x}{r_i}\right)^{-j} = \frac{C_{i,j}}{(-r_i)^j} \cdot \frac{1}{r_i^n} \cdot \binom{n+j-1}{n}$$

to obtain

$$[x^n]\frac{C_0}{x-p} = \frac{C_0}{-p} \cdot \frac{1}{p^n} = \frac{1}{p^n \sum_{k \in S} kp^k}$$

and

$$[x^n]\sum_i \left(\frac{C_{i,1}}{x-r_i} + \frac{C_{i,2}}{(x-r_i)^2} + \cdots + \frac{C_{i,k_i}}{(x-r_i)^{k_i}}\right) = O\left(\frac{n^{k_1-1}}{|r_1|^n}\right).$$

Taking into account the fact that $k_1 \leq M$, we have

$$|\Lambda_n| = \frac{1}{p^n \sum_{k \in S} kp^k} + O\left(\frac{n^{M-1}}{r^n}\right). \qquad \square$$

For future reference, we record the following corollary which is derived easily from Theorem 2.

Corollary 1. *For any real* $q \in (p, r)$,

$$\frac{1}{|\Lambda_n|} = p^n \sum_{k \in S} kp^k\left(1 + O\left(\left(\frac{p}{q}\right)^n\right)\right).$$

3. MOMENTS OF LOG B

In this section we combine generating function identities from [**31, 32**] with singularity analysis [**10, 11**] to estimate the moments of $\log \mathbf{B}$.

Recall that r is the magnitude of the second smallest root of f, that $0 < p < 1$, and that $p < r$.

Theorem 3. *There exist constants $a_1 > 0$ and a_0 such that, for any given $q \in (p, \min(1, r))$, the expected value of the log product of parts is*

$$E_n(\log \mathbf{B}) = a_1 n + a_0 + O\left(n \left(\frac{p}{q} \right)^n \right).$$

Proof. Define the moment generating function for the random variable $\log \mathbf{B}$,

$$M_n(t) = E_n(e^{t \log \mathbf{B}}) = E_n(\mathbf{B}^t).$$

By Lemma 2, we have

$$E_n(\mathbf{B}^t) = \frac{1}{|\Lambda_n|} [x^n] \frac{1}{F(x, t)}.$$

Hence

$$E_n(\log \mathbf{B}) = M_n'(0) = \frac{1}{|\Lambda_n|} \frac{\mathrm{d}}{\mathrm{d}t} [x^n] \frac{1}{F(x, t)} \bigg|_{t=0}.$$

Since $\dfrac{1}{F(x, t)}$ is analytic throughout the disk $|x| < p$, we know that it has a Taylor series representation

$$\frac{1}{F(x, t)} = \sum_{n=0}^{\infty} a_n x^n \quad \text{where} \quad a_n = \frac{1}{2\pi i} \int_C \frac{\frac{1}{F(x, t)}}{x^{n+1}} \, \mathrm{d}x$$

for a suitable contour C. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t} [x^n] \frac{1}{F(x, t)} = \frac{\mathrm{d}}{\mathrm{d}t} a_n = \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2\pi i} \int_C \frac{\frac{1}{F(x, t)}}{x^{n+1}} \, \mathrm{d}x.$$

Since $\dfrac{1}{F(x, t)}$ and its partial derivative with respect to t are both analytic throughout $|x| < p$, we can move the derivative inside the integral sign to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2\pi i} \int_C \frac{\frac{1}{F(x, t)}}{x^{n+1}} \, \mathrm{d}x = \frac{1}{2\pi i} \int_C \frac{\frac{\partial}{\partial t} \frac{1}{F(x, t)}}{x^{n+1}} \, \mathrm{d}x = [x^n] \frac{\partial}{\partial t} \frac{1}{F(x, t)}.$$

Calculating the partial derivative and evaluating at $t = 0$, we see that

$$M_n'(0) = \frac{1}{|\Lambda_n|} [x^n] \frac{\sum_{k \in S} (\log k) x^k}{\left(1 - \sum_{k \in S} x^k \right)^2}.$$

To simplify this expression, we define $L_i(x) = \sum\limits_{k \in S} (\log k)^i x^k$ and $D(x) = \dfrac{f(x)}{x - p}$ and note that D has no zeroes in the disk $|x| < r$. Then

$$(2) \qquad M_n'(0) = \frac{1}{|\Lambda_n|}[x^n]\frac{L_1(x)}{(x - p)^2 D(x)^2}.$$

To estimate the right side of (2), note that the function $G_0(x) = \dfrac{L_1(x)}{D(x)^2}$ is analytic in the disk $|x| \le q$ for any $q < \min(1, r)$. If we expand $G_0(x)$ around p, we have

$$G_0(x) = G_0(p) + G_0'(p)(x - p) + \tilde{G}_0(x)$$

where $\tilde{G}_0(x) = \sum\limits_{k=2}^{\infty} \dfrac{G_0^{(k)}(p)}{k!}(x - p)^k$. Therefore

$$(3) \qquad [x^n]\frac{G_0(x)}{(x - p)^2} = [x^n]\frac{G_0(p)}{(x - p)^2} + [x^n]\frac{G_0'(p)}{(x - p)} + [x^n]\frac{\tilde{G}_0(x)}{(x - p)^2}.$$

We use (1) to obtain

$$[x^n]\frac{G_0(p)}{(x - p)^2} = \frac{G_0(p)(n + 1)}{p^2 \cdot p^n} \quad \text{and} \quad [x^n]\frac{G_0'(p)}{(x - p)} = \frac{G_0'(p)}{-p \cdot p^n}.$$

To bound the last term in (3), note that $\dfrac{\tilde{G}_0(x)}{(x - p)^2}$ is also analytic in the disk $|x| \le q$ (with a removable singularity at $x = p$). Choosing γ to be a positively-oriented circle of radius q, centered at the origin, we use Cauchy's inequality to get

$$\left|[x^n]\frac{\tilde{G}_0(x)}{(x - p)^2}\right| = \left|\frac{1}{2\pi i}\int_\gamma \frac{\tilde{G}_0(x)}{(x - p)^2 x^{n+1}}\ \mathrm{d}x\right| \le \frac{\max\left|\frac{\tilde{G}_0(x)}{(x - p)^2}\right|}{q^n} = O\left(\frac{1}{q^n}\right).$$

Combining this with (3) and Corollary 1, with the value of q chosen in $(p, \min(1, r))$, we get

$$E_n(\log \mathbf{B}) = \frac{1}{|\Lambda_n|}[x^n]\frac{G_0(x)}{(x - p)^2}$$

$$= \frac{G_0(p)\sum\limits_{k \in S} kp^k(n + 1)}{p^2} + \frac{G_0'(p)\sum\limits_{k \in S} kp^k}{-p} + O\left(n\left(\frac{p}{q}\right)^n\right).$$

Using the fact that $D^{(k)}(p) = \dfrac{f^{(k+1)}(p)}{k + 1}$, we can evaluate the constants $G_0(p)$ and $G_0'(p)$ to obtain the statement of the theorem.

Theorem 4. *There exist constants $b_1 > 0$ and b_0 such that, for any given $q \in (p, \min(1, r))$, the variance of the log product of parts is*

$$V_n(\log \mathbf{B}) = b_1 n + b_0 + O\left(n^2 \left(\frac{p}{q}\right)^n\right).$$

Proof. By the same method as above, we obtain

$$M_n''(0) = \frac{1}{|\Lambda_n|}[x^n]\left(\frac{2\left(\sum_{k \in S}(\log k)x^k\right)^2}{\left(1 - \sum_{k \in S}x^k\right)^3} + \frac{\sum_{k \in S}(\log k)^2 x^k}{\left(1 - \sum_{k \in S}x^k\right)^2}\right)$$

$$= \frac{1}{|\Lambda_n|}[x^n]\left(\frac{2L_1(x)^2}{(x - p)^3 D(x)^3} + \frac{L_2(x)}{(x - p)^2 D(x)^2}\right).$$

Since the functions $G_1(x) = \dfrac{L_1(x)^2}{D(x)^3}$ and $G_2(x) = \dfrac{L_2(x)}{D(x)^2}$ are both analytic in the disk $|x| \le q < \min(1, r)$, we can expand them around $x = p$ as in the previous proof. Hence, for the first term above, we have

$$[x^n]\frac{G_1(x)}{(x - p)^3} = [x^n]\frac{G_1(p)}{(x - p)^3} + [x^n]\frac{G_1'(p)}{(x - p)^2} + [x^n]\frac{G_1''(p)}{2(x - p)} + O\left(\frac{1}{q^n}\right)$$

$$= \frac{G_1(p)(n + 1)(n + 2)}{-2p^3 \cdot p^n} + \frac{G_1'(p)(n + 1)}{p^2 \cdot p^n} + \frac{G_1''(p)}{-2p \cdot p^n} + O\left(\frac{1}{q^n}\right).$$

Similarly, for the second term above, we have

$$[x^n]\frac{G_2(x)}{(x - p)^2} = [x^n]\frac{G_2(p)}{(x - p)^2} + [x^n]\frac{G_2'(p)}{(x - p)} + O\left(\frac{1}{q^n}\right)$$

$$= \frac{G_2(p)(n + 1)}{p^2 \cdot p^n} + \frac{G_2'(p)}{-p \cdot p^n} + O\left(\frac{1}{q^n}\right).$$

Now by Corollary 1 we get

$$E_n((\log \mathbf{B})^2) = \frac{1}{|\Lambda_n|}[x^n]\frac{2G_1(x)}{(x - p)^3} + \frac{1}{|\Lambda_n|}[x^n]\frac{G_2(x)}{(x - p)^2}$$

$$= \frac{G_1(p)\sum_{k \in S}kp^k(n + 1)(n + 2)}{-p^3} + \frac{2G_1'(p)\sum_{k \in S}kp^k(n + 1)}{p^2}$$

$$+ \frac{G_1''(p)\sum_{k \in S}kp^k}{-p} + \frac{G_2(p)\sum_{k \in S}kp^k(n + 1)}{p^2} + \frac{G_2'(p)\sum_{k \in S}kp^k}{-p} + O\left(n^2\left(\frac{p}{q}\right)^n\right).$$

These constants can be evaluated as in the previous proof, and $E_n(\log \mathbf{B})^2$ can be calculated using Theorem 3, to obtain the statement of the theorem. $\square$

In theory, any moment of $\log \mathbf{B}$ can be calculated using the methods above. However there is a major problem with using this method to calculate central moments: There are many cross terms, and it becomes unmanageable to do the calculations by hand. Due to the length and messiness of the calculations, we record the following result without proof. More details can be found in [**32**].

Theorem 5. *Define R_n to be the fourth central moment with respect to P_n. Then*

$$R_n(\log \mathbf{B}) = E_n((\log \mathbf{B} - E_n(\log \mathbf{B}))^4) = O(n^2).$$

An anonymous referee pointed out an alternative approach, related to the ideas in [**23**], for calculating the central moments of $\log \mathbf{B}$: The generating function $T_m(x) = \sum\limits_n |\Lambda_n| E_n(\log \mathbf{B} - \mu n)^m x^n$ satisfies the recurrence

$$T_m(x) = \frac{1}{1 - S_0(x)} \sum_{k=1}^{m} \binom{m}{k} T_{m-k}(x) S_k,$$

where

$$S_k(x) = \sum_{j \in S} (\log j - \mu j)^k x^j.$$

The referee believes that the m'th central moment of $\log \mathbf{B}$ can then be estimated using this recurrence and an induction argument. He remarked that, by the method of moments, this would give an alternative proof that $\log \mathbf{B}$ is asymptotically normal, not only for cofinite S but also for more general sets [**12**]. One disadvantage of the referee's approach is that it does not yield an explicit error term like that in Theorem 1; however, it would certainly allow the lengthy calculations of Theorems 3, 4, and 5 to be bypassed.

4. METHOD OF CONCATENATED COMPOSITIONS

In this section we present a method for breaking down a composition consisting of a random number of parts into a sequence consisting of a deterministic number of subcompositions (of approximately the same size). The approach is stylistically similar to Bernstein's blocking method, which separates a sequence of dependent random variables into an alternating sequence of "large blocks" and "small blocks." Before giving a precise, notation-laden version of the technique, we give an informal description in terms of the classical bijection between compositions of n and sequences of n balls colored white or black with the last ball black. (A composition $\vec{\lambda} = (\lambda_1, \lambda_2, \dots)$ corresponds to the sequence of n colored balls in which the position of the i'th black ball is $\sum\limits_{k=1}^{i} \lambda_k$.)

A method that does not quite work is the following: form a sequence of $m+1$ compositions by using the first $\left\lfloor \dfrac{n}{m} \right\rfloor$ balls as the first composition, the second $\left\lfloor \dfrac{n}{m} \right\rfloor$ balls as the second composition, etc. The main problem with this approach is that

the ball in position $k\left\lfloor\dfrac{n}{m}\right\rfloor$ need not be colored black, and consequently the k'th sequence of $\left\lfloor\dfrac{n}{m}\right\rfloor$ balls need not correspond to a composition.

To salvage this idea, we make use of a simple observation: there must be *some* part of $\vec{\lambda}$ corresponding to the ball in position $k\left\lfloor\dfrac{n}{m}\right\rfloor$. Let Π_0 be the composition that is formed by selecting these m regularly spaced parts of $\vec{\lambda}$. The kth part of Π_0 is the part of $\vec{\lambda}$ that contains the ball in position $k\left\lfloor\dfrac{n}{m}\right\rfloor$. Then the parts of Π_0 form natural dividers for decomposing $\vec{\lambda}$. As an example, suppose $m = 4$ and consider the composition $\vec{\lambda} = (3, 2, 3, 1, 2, 2, 2, 3, 2, 2, 2, 1)$ that corresponds to the sequence shown here:

○ ○ ● ○ ● ○ ○ ● ● ○ ● ○ ● ○ ● ○ ○ ● ○ ● ○ ● ○ ● ●

Note that $\left\lfloor\dfrac{n}{m}\right\rfloor = \left\lfloor\dfrac{25}{4}\right\rfloor = 6$. The balls at positions 6, 12, 18, and 24 (marked below with arrows) belong to parts of $\vec{\lambda}$ with respective sizes 3, 2, 3, and 2. Therefore $\Pi_0 = (3, 2, 3, 2)$. Circled below are the balls that correspond to the parts of Π_0:

○ ○ ● ○ ● ◎ ◎ ◉ ● ○ ● ◎ ◉ ○ ● ◎ ◎ ◉ ○ ● ○ ● ◎ ◉ ●
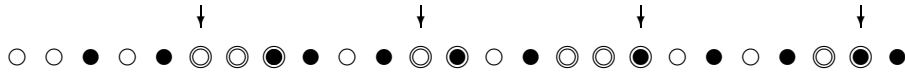
If we remove all the balls that correspond to the parts of Π_0, then we are left with a sequence of five compositions $\Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5$ (of various sizes) as shown here:

○ ○ ● ○ ●   ● ○ ●   ○ ●   ○ ● ○ ●   ●
Π_1 Π_2 Π_3 Π_4 Π_5
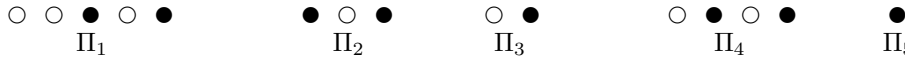
In order for this decomposition to be well-defined, it is necessary to bound the sizes of the parts. (Consider what happens if $\vec{\lambda}$ is a single part of size n.)

We now proceed with a more formal specification of the decomposition process. For any positive integers β and n, let Λ_n^β denote the set of compositions of n whose parts are all in $[1, \beta] \cap S$. Let m be a positive integer such that $\left\lfloor\dfrac{n}{m}\right\rfloor > 2\beta$. For each $\vec{\lambda}$ in Λ_n^β and for $i = 1, \ldots, m$, define

$$\tau_i = \min\left\{t : \sum_{k=1}^{t} \lambda_k \geq i\left\lfloor\dfrac{n}{m}\right\rfloor\right\}.$$

Let τ be the total number of parts of the composition $\vec{\lambda}$. In our example, $\tau_1 = 3$, $\tau_2 = 6$, $\tau_3 = 8$, $\tau_4 = 11$, and $\tau = 12$. Define the following compositions:

$$\Pi_0 = \langle\lambda_{\tau_j}\rangle_{j=1}^m \quad \Pi_1 = \langle\lambda_j\rangle_{j=1}^{\tau_1-1} \quad \Pi_i = \langle\lambda_j\rangle_{j=\tau_{i-1}+1}^{\tau_i-1} \text{ for } i = 2 \leq i \leq m.$$

If n is not a multiple of m, define $\Pi_{m+1} = \langle\lambda_j\rangle_{j=\tau_m+1}^\tau$. If n is a multiple of m, then $\tau_m = n$ and we do not need an $(m+1)$'st composition. However, it will simplify the

presentation if we adopt the convention, when n is a multiple of m, that Π_{m+1} is an "empty composition of zero with no parts" and that $\mathbf{B}(\Pi_{m+1}) = 1$. This completes our definition of what it means to decompose a composition in Λ_n^β when $m \in \mathbb{Z}_+$ and $\left\lfloor \dfrac{n}{m} \right\rfloor > 2\beta$.

As a byproduct of the decomposition process, we have a natural way to partition Λ_n^β. This is important for our proof, because it enables us to write $\log \mathbf{B}$ as a sum of conditionally independent random variables. Let $p_1 = 1$, and for $2 \le i \le m+1$ let $p_i = 1 + \sum_{k=1}^{\tau_i - 1} \lambda_k$ denote the position of the first ball that corresponds to Π_i. Define $W_i = (\pi_i, p_i)$, where $\pi_i = |\Pi_i|$ is the number that Π_i composes. Finally, let $\vec{W} = (W_1, \dots, W_{m+1})$. Note that $\vec{W}$ is determined by $\vec{\lambda}$, but that many compositions correspond to a given choice of $\vec{W}$. Define an equivalence relation on Λ_n^β by declaring two compositions to be equivalent if and only if they determine the same $\vec{W}$. Let $\Lambda_{\vec{W}}$ be the equivalence class of compositions in Λ_n^β that correspond to a given choice of $\vec{W}$, and let $\mathcal{W}_n$ be the set of equivalence classes.

Now observe that

$$(4) \qquad \log \mathbf{B}(\vec{\lambda}) = \sum_{i=0}^{m+1} \log \mathbf{B}(\Pi_i(\vec{\lambda})).$$

The next theorem says that the random variables $\mathbf{L}_i = \log \mathbf{B}(\Pi_i(\vec{\lambda}))$ are conditionally independent given $\vec{W}$.

Theorem 6. *If Q_n is the uniform probability measure on Λ_n^β, then for all $\vec{W}$ and all $y_1, y_2, \dots, y_{m+1}$,*

$$Q_n\left(\forall i \ \mathbf{L}_i = y_i | \vec{W}\right) = \prod_{i=1}^{m+1} Q_n(\mathbf{L}_i = y_i | \vec{W}).$$

Proof. For each choice of $\vec{W}$, there is an obvious bijection $\Psi_{\vec{W}}$ from $\Lambda_{\vec{W}}$ onto the product $\Lambda_{\pi_1}^\beta \times \Lambda_{\pi_2}^\beta \cdots \times \Lambda_{\pi_{m+1}}^\beta$, namely $\Psi_{\vec{W}}(\vec{\lambda}) = (\Pi_1, \dots, \Pi_{m+1})$. Hence

$$(5) \qquad |\Lambda_{\vec{W}}| = \prod_{i=1}^{m+1} |\Lambda_{\pi_i}^\beta|.$$

For any choice of $y_1, \dots, y_{m+1}$, we have

$$Q_n\left(\forall i \ \mathbf{L}_i = y_i | \vec{W}\right) = \frac{|\{\vec{\lambda} \in \Lambda_{\vec{W}} : \forall i \ \mathbf{L}_i = y_i\}|}{|\Lambda_{\vec{W}}|}$$

$$= \frac{|\{(\Pi_1, \dots, \Pi_{m+1}) : \forall i \ \mathbf{L}_i = y_i\}|}{|\Lambda_{\vec{W}}|} = \frac{\prod_{i=1}^{m+1} |\{\Pi_i : \mathbf{L}_i = y_i\}|}{|\Lambda_{\vec{W}}|}.$$

Recalling (5) then multiplying by a factor of 1, we get

$$
Q_n\left(\forall i\ \mathbf{L}_i = y_i | \vec{W}\right) = \prod_{i=1}^{m+1} \frac{|\{\Pi_i : \mathbf{L}_i = y_i\}| \cdot \prod_{j \neq i} |\Lambda^\beta_{\pi_j}|}{|\Lambda^\beta_{\pi_i}| \cdot \prod_{j \neq i} |\Lambda^\beta_{\pi_j}|}
$$

$$
= \prod_{i=1}^{m+1} \frac{|\{\vec{\lambda} \in \Lambda_{\vec{W}} : \mathbf{L}_i = y_i\}|}{|\Lambda_{\vec{W}}|} = \prod_{i=1}^{m+1} Q_n(\mathbf{L}_i = y_i | \vec{W}). \qquad \square
$$

Although we do not need it, it is worth mentioning a stronger statement that is perhaps more intuitive. The following are equivalent methods for picking a random composition:

Method 1. Pick a composition $\vec{\lambda}$ from a uniform distribution on the set of all compositions in Λ^β_n with a given $\vec{W}$.

Method 2. For each $i \geq 1$, independently pick Π_i. The numbers π_i are determined by $\vec{W}$, and each Π_i is chosen from a uniform distribution on the set of compositions in $\Lambda^\beta_{\pi_i}$. Then concatenate $\Pi_1, \Pi_2, \ldots \Pi_{m+1}$, using the parts of Π_0 as dividers, to form a composition $\vec{\lambda}$.

Finally, for future reference we state a simple lemma that is obvious from the construction in this section and can be proved using calculations similar to those in the proof of Theorem 6.

Lemma 3. *Consider $\mathbf{L}_i = \log \mathbf{B}(\Pi_i(\vec{\lambda}))$ as a random variable on Λ^β_n with respect to the conditional probability measure $P_n(\cdot | \Lambda^\beta_n)$. Assume β is chosen in such a way that $\vec{W}$ is well-defined and $\Lambda_{\vec{W}} \neq \emptyset$. If $i \leq m$, then for any choice of t,*

$$
E_n(\mathbf{L}_i^t | \Lambda^\beta_n, \vec{W}) = E_{\pi_i}((\log \mathbf{B})^t | \Lambda^\beta_n).
$$

5. COMPARING CONDITIONAL DISTRIBUTIONS

Recall that $\log \mathbf{B} = \sum_{i=0}^{m+1} \mathbf{L}_i$. Reasoning heuristically, one might expect Theorem 1 to be a consequence of the central limit theorem. There are two problems with this idea.

First, the random variables $\mathbf{L}_i$ are defined on Λ^β_n, not Λ_n; it does not make sense to talk about the probability measure P_n in reference to $\mathbf{L}_i$. This problem is only a minor technicality because most compositions have no large parts, and we can, without loss of generality, reduce to the case of compositions selected from Λ^β_n using the uniform probability measure $Q_n(\cdot) = P_n(\cdot | \Lambda^\beta_n)$.

The second problem is that the random variables $\mathbf{L}_i$ are not independent with respect to the probability measure $Q_n(\cdot)$. However, we can use Theorem 6 to obtain

the necessary conditions. Conditioning on our choice of $\vec{W}$, we have

$$(6) \quad P_n\left(\frac{\log \mathbf{B} - \mu_n}{\sigma_n} \leq x \,\Big|\, \Lambda_n^\beta\right) = \sum_{\vec{W} \in \mathcal{W}_n} P_n(\vec{W}) P_n\left(\frac{\sum_{i=0}^{m+1} \mathbf{L}_i - \mu_n}{\sigma_n} \leq x \,\Big|\, \Lambda_n^\beta, \vec{W}\right).$$

The central limit theorem can be applied $|\mathcal{W}_n|$ times: Once for each of the terms in the right hand side of (6), using a different probability measure $P_n(\cdot|\Lambda_n^\beta, \vec{W})$ each time. This may not seem like a promising approach, since a mixture of normal distributions is not necessarily normal. However, in our case, the normal distributions all have approximately the same mean and variance, so we do in fact get the desired result.

The preceding paragraphs contain the main idea of the proof. However the reasoning is necessarily vague and incomplete. The remainder of this section contains a series of elementary lemmas that are needed before a mathematically sound version of the argument can be completed in the next section.

We need a precise statement of the fact that a typical composition has no large parts. The following crude first moment estimate is convenient for our purposes; more in-depth studies have been carried out by others, leading to more precise [28] and more general [3] results. First we set the parameters m and β:

$$(7) \qquad m = \left\lfloor \frac{n^{1/3}}{(5 \log_{1/p} n)^{2/3}} \right\rfloor \qquad \beta = \lfloor 5 \log_{1/p} n \rfloor.$$

Lemma 4. *Let Λ_n^β be the event that an S-restricted composition of n has no parts of size larger than β. Then*

$$P_n(\overline{\Lambda_n^\beta}) = O(np^\beta).$$

Proof. Let $A_{i,j}$ be the event that a part of size j begins in position i. Then

$$\overline{\Lambda_n^\beta} = \bigcup_{i=1}^n \bigcup_{j>\beta} A_{i,j}.$$

Compositions in $A_{i,j}$ are constructed by first choosing a composition of $i-1$, then appending a part of of size j, then appending a composition of $n-(i-1)-j$. If we adopt the convention that $|\Lambda_0| = 1$, then for all i and j

$$P_n(A_{i,j}) = \frac{|\Lambda_{i-1}| \cdot |\Lambda_{n-i-j+1}|}{|\Lambda_n|}.$$

Using first Boole's inequality and then Theorem 2 and (7), we get

$$P_n(\overline{\Lambda_n^\beta}) \leq \sum_{i=1}^n \sum_{\beta<j\leq n} \frac{|\Lambda_{i-1}| \cdot |\Lambda_{n-i-j+1}|}{|\Lambda_n|} = O(np^\beta). \qquad \square$$

The next lemma says that the moments are only slightly perturbed if we impose a reasonable bound β on the sizes of the parts.

Lemma 5. *For any t and any choice of β,*

$$E_n((\log \mathbf{B})^t) = E_n((\log \mathbf{B})^t | \Lambda_n^\beta) + O(n^{t+1} p^\beta).$$

Proof. The largest value that $\log \mathbf{B}$ can possibly attain is $O(n)$. Therefore, by Lemma 4, we have

$$E_n((\log \mathbf{B})^t) = E_n((\log \mathbf{B})^t | \Lambda_n^\beta) P_n(\Lambda_n^\beta) + E_n((\log \mathbf{B})^t | \overline{\Lambda_n^\beta}) P_n(\overline{\Lambda_n^\beta})$$

$$\leq E_n((\log \mathbf{B})^t | \Lambda_n^\beta) + \max((\log \mathbf{B})^t) P_n(\overline{\Lambda_n^\beta})$$

$$= E_n((\log \mathbf{B})^t | \Lambda_n^\beta) + O(n^t) \cdot O(np^\beta) = E_n((\log \mathbf{B})^t | \Lambda_n^\beta) + O(n^{t+1} p^\beta).$$

By a similar argument, $E_n((\log \mathbf{B})^t) \geq E_n((\log \mathbf{B})^t | \Lambda_n^\beta) - O(n^{t+1} p^\beta)$. $\qquad\square$

For future reference, we note the following immediate corollary of Lemma 5.

Corollary 2. *The mean, variance, and fourth moment of $\log \mathbf{B}$ are respectively*

$$E_n(\log \mathbf{B}) = E_n(\log \mathbf{B} | \Lambda_n^\beta) + O(n^2 p^\beta)$$

$$V_n(\log \mathbf{B}) = V_n(\log \mathbf{B} | \Lambda_n^\beta) + O(n^3 p^\beta)$$

$$R_n(\log \mathbf{B}) = R_n(\log \mathbf{B} | \Lambda_n^\beta) + O(n^5 p^\beta).$$

As a direct consequence of Lemma 3 and Corollary 2, we note the following.

Corollary 3. *For $i = 1, \ldots, m$,*

$$E_n(\mathbf{L}_i | \Lambda_n^\beta, \vec{W}) = E_{\pi_i}(\log \mathbf{B}) + O(\pi_i^2 p^\beta)$$

$$V_n(\mathbf{L}_i | \Lambda_n^\beta, \vec{W}) = V_{\pi_i}(\log \mathbf{B}) + O(\pi_i^3 p^\beta)$$

$$R_n(\mathbf{L}_i | \Lambda_n^\beta, \vec{W}) = R_{\pi_i}(\log \mathbf{B}) + O(\pi_i^5 p^\beta).$$

Corollary 4.

$$\sum_{i=1}^m E_n(\mathbf{L}_i | \Lambda_n^\beta, \vec{W}) = E_n(\log \mathbf{B}) + O(m\beta)$$

$$\sum_{i=1}^m V_n(\mathbf{L}_i | \Lambda_n^\beta, \vec{W}) = V_n(\log \mathbf{B}) + O(m\beta).$$

Proof. By Corollary 3 and Theorem 3, we have

$$E_n(\mathbf{L}_i | \Lambda_n^\beta, \vec{W}) = E_{\pi_i}(\log \mathbf{B}) + O(\pi_i^2 p^\beta) = a_1 \pi_i + a_0 + O\left(\pi_i \left(\frac{p}{q}\right)^{\pi_i}\right) + O(\pi_i^2 p^\beta).$$

Note that, for $i = 1, \ldots, m$,

$$(8) \qquad \frac{n}{m} - 2\beta \leq \pi_i \leq \frac{n}{m}.$$

Therefore

$$E_n(\mathbf{L}_i | \Lambda_n^\beta, \vec{W}) = a_1\left(\frac{n}{m} + O(\beta)\right) + a_0 + O\left(\frac{n}{m}\left(\frac{p}{q}\right)^{\frac{n}{m}-2\beta}\right) + O\left(\left(\frac{n}{m}\right)^2 p^\beta\right).$$

Noting the definitions of m and β in (7), we have

$$\sum_{i=1}^{m} E_n(\mathbf{L}_i | \Lambda_n^\beta, \vec{W}) = a_1 n + O(m\beta).$$

We make a similar calculation for the variance, using Theorem 4:

$$V_n(\mathbf{L}_i | \Lambda_n^\beta, \vec{W}) = V_{\pi_i}(\log \mathbf{B}) + O(\pi_i^3 p^\beta) = b_1 \pi_i + b_0 + O\left(\pi_i^2 \left(\frac{p}{q}\right)^{\pi_i}\right) + O\left(\pi_i^3 p^\beta\right)$$

$$= b_1\left(\frac{n}{m} + O(\beta)\right) + b_0 + O\left(\left(\frac{n}{m}\right)^2 \left(\frac{p}{q}\right)^{\frac{n}{m}-2\beta}\right) + O\left(\left(\frac{n}{m}\right)^3 p^\beta\right).$$

Noting (7), we have

$$\sum_{i=1}^{m} V_n(\mathbf{L}_i | \Lambda_n^\beta, \vec{W}) = b_1 n + O(m\beta).$$

Theorems 3 and 4 give the statement of the corollary. $\qquad\qquad\square$

The following two theorems are very well-known. For example, in [**9**], see page 544 for Theorem 7 and page 155 for Theorem 8. We use $\Phi(x)$ to denote the standard normal distribution.

Theorem 7 (Esseen inequality)**.** *There is a positive constant A such that, for any choice of mutually independent* (*not necessarily identically distributed*) *random variables $X_1, \ldots, X_m$, if $E(X_i) = 0$ and $E(|X_i|)^3 < \infty$ for $i = 1, \ldots, m$, then*

$$\sup_x \left| P\left(\frac{\sum_{i=1}^{m} X_i}{\sqrt{\sum_{i=1}^{m} E(X_i^2)}} < x\right) - \Phi(x) \right| \leq \frac{A\sum_{i=1}^{m} E(|X_i|^3)}{\left(\sum_{i=1}^{m} E(X_i^2)\right)^{3/2}}.$$

Theorem 8. *If a random variable X has a moment of order s, then for positive $r \leq s$,*

$$E(|X|^r)^{1/r} \leq E(|X|^s)^{1/s}.$$

Corollary 5.

$$\sum_{i=1}^{m} E_n(|\mathbf{L}_i - E_n(\mathbf{L}_i|\Lambda_n^\beta, \vec{W})|^3 \ |\Lambda_n^\beta, \vec{W}) = O\Big(\frac{n^{3/2}}{m^{1/2}}\Big).$$

Proof. By Theorem 8, we have

$$E_n(|\mathbf{L}_i - E_n(\mathbf{L}_i|\Lambda_n^\beta, \vec{W})|^3|\Lambda_n^\beta, \vec{W}) \le R_n(\mathbf{L}_i|\Lambda_n^\beta, \vec{W})^{3/4}.$$

Applying Corollary 3 followed by Theorem 5, we have

$$\sum_{i=1}^{m} E_n(|\mathbf{L}_i - E_n(\mathbf{L}_i|\Lambda_n^\beta, \vec{W})|^3|\Lambda_n^\beta, \vec{W}) \le \sum_{i=1}^{m} R_n(\mathbf{L}_i|\Lambda_n^\beta, \vec{W})^{3/4}$$

$$= \sum_{i=1}^{m} \big(R_{\pi_i}(\log \mathbf{B}) + O(\pi_i^5 p^\beta)\big)^{3/4} = \sum_{i=1}^{m} \big(O(\pi_i^2) + O(\pi_i^5 p^\beta)\big)^{3/4}.$$

Recalling (8) and noting (7), the righthand side becomes

$$\sum_{i=1}^{m} O\left(\frac{n^2}{m^2}\right)^{3/4} = \sum_{i=1}^{m} O\Big(\frac{n^{3/2}}{m^{3/2}}\Big) = O\Big(\frac{n^{3/2}}{m^{1/2}}\Big).$$

6. ASYMPTOTIC LOGNORMALITY OF B

Proof of Theorem 1. We will use the following shorthand notation:

$$\mu_n = E_n(\log \mathbf{B}), \quad e_n = \sum_{i=1}^{m} E_n(\mathbf{L}_i|\Lambda_n^\beta, \vec{W}), \quad \sigma_n^2 = V_n(\log \mathbf{B}),$$

$$v_n = \sum_{i=1}^{m} V_n(\mathbf{L}_i|\Lambda_n^\beta, \vec{W}), \quad t_n = \sum_{i=1}^{m} E_n(|\mathbf{L}_i - E_n(\mathbf{L}_i|\Lambda_n^\beta, \vec{W})|^3|\Lambda_n^\beta, \vec{W}).$$

By Lemma 4, Corollary 4, and Corollary 5, we have

$$(9) \qquad\qquad P_n(\overline{\Lambda_n^\beta}) = O\Big(\frac{1}{n^4}\Big)$$

$$(10) \qquad\qquad \mu_n = e_n + O((n \log n)^{1/3})$$

$$(11) \qquad\qquad \sigma_n^2 = v_n + O((n \log n)^{1/3})$$

$$(12) \qquad\qquad t_n = O(n^{4/3}(\log n)^{1/3}).$$

We begin the calculation by using (9) to obtain

$$(13) \qquad P_n\left(\frac{\log \mathbf{B} - \mu_n}{\sigma_n} \le x\right) = P_n\left(\frac{\log \mathbf{B} - \mu_n}{\sigma_n} \le x|\Lambda_n^\beta\right) P_n(\Lambda_n^\beta)$$

$$+ P_n\left(\frac{\log \mathbf{B} - \mu_n}{\sigma_n} \le x|\overline{\Lambda_n^\beta}\right) P_n(\overline{\Lambda_n^\beta})$$

$$= P_n\left(\frac{\log \mathbf{B} - \mu_n}{\sigma_n} \le x|\Lambda_n^\beta\right) + O\left(\frac{1}{n^4}\right).$$

We recall that, for a composition in Λ_n^β,

$$\log \mathbf{B} = \sum_{i=0}^{m+1} \mathbf{L}_i.$$

Conditioning on our choice of $\vec{W}$, we have

$$(14) \quad P_n\left(\frac{\log \mathbf{B} - \mu_n}{\sigma_n} \le x | \Lambda_n^\beta\right) = \sum_{\vec{W} \in \mathcal{W}_n} P_n(\vec{W}) P_n\left(\frac{\sum_{i=0}^{m+1} \mathbf{L}_i - \mu_n}{\sigma_n} \le x | \Lambda_n^\beta, \vec{W}\right).$$

Using (10) and (11), and also noting that $\mathbf{L}_0 \le m \log \beta < m\beta \le (n \log n)^{1/3}$ and $\mathbf{L}_{m+1} \le m \log \beta < m\beta \le (n \log n)^{1/3}$, we have

$$\sum_{i=0}^{m+1} \mathbf{L}_i - \mu_n = \sum_{i=1}^{m} \mathbf{L}_i - e_n + O((n \log n)^{1/3}) \quad \text{and} \quad \sigma_n = \sqrt{v_n + O((n \log n)^{1/3})}.$$

Therefore,

$$(15) \quad P_n\left(\frac{\sum_{i=0}^{m+1} \mathbf{L}_i - \mu_n}{\sigma_n} \le x | \Lambda_n^\beta, \vec{W}\right)$$

$$= P_n\left(\frac{\sum_{i=1}^{m} \mathbf{L}_i - e_n + O((n \log n)^{1/3})}{\sqrt{v_n + O((n \log n)^{1/3})}} \le x | \Lambda_n^\beta, \vec{W}\right) = P_n\left(\frac{\sum_{i=1}^{m} \mathbf{L}_i - e_n}{\sqrt{v_n}} \le s_{n,x} | \Lambda_n^\beta, \vec{W}\right)$$

where $s_{n,x} = \left(x - \frac{O((n \log n)^{1/3})}{\sqrt{v_n + O((n \log n)^{1/3})}}\right)\sqrt{1 + \frac{O((n \log n)^{1/3})}{v_n}}$. We can now apply to (15) the Esseen inequality from Theorem 7, followed by (12) and Theorem 4 to obtain

$$(16) \quad P_n\left(\frac{\sum_{i=1}^{m} \mathbf{L}_i - e_n}{\sqrt{v_n}} \le s_{n,x} | \Lambda_n^\beta, \vec{W}\right) = \Phi(s_{n,x}) + O\left(\frac{t_n}{(v_n)^{3/2}}\right)$$

$$= \Phi(s_{n,x}) + O\left(\frac{(\log n)^{1/3}}{n^{1/6}}\right).$$

Next we note the approximation

$$s_{n,x} = x\left(1 + O\left(\frac{(\log n)^{1/3}}{n^{2/3}}\right)\right) + O\left(\frac{(\log n)^{1/3}}{n^{1/6}}\right).$$

If x is positive, then there exist positive constants c_1 and c_2 such that, for any n,

$$x\left(1 - \frac{c_1(\log n)^{1/3}}{n^{2/3}}\right) - \frac{c_2(\log n)^{1/3}}{n^{1/6}} \leq s_{n,x} \leq x\left(1 + \frac{c_1(\log n)^{1/3}}{n^{2/3}}\right) + \frac{c_2(\log n)^{1/3}}{n^{1/6}}.$$

Since $0 \leq e^{-t^2/2} \leq 1$,

$$\Phi(s_{n,x}) \leq \Phi(x) + \frac{1}{\sqrt{2\pi}} \int_x^{x\left(1+\frac{c_1(\log n)^{1/3}}{n^{2/3}}\right)+\frac{c_2(\log n)^{1/3}}{n^{1/6}}} e^{-t^2/2} \, dt$$

$$\leq \Phi(x) + \left(x\left(1 + \frac{c_1(\log n)^{1/3}}{n^{2/3}}\right) + \frac{c_2(\log n)^{1/3}}{n^{1/6}} - x\right)$$

$$= \Phi(x) + O\left(\frac{x(\log n)^{1/3}}{n^{2/3}} + \frac{(\log n)^{1/3}}{n^{1/6}}\right).$$

Similarly, $\Phi(s_{n,x}) \geq \Phi(x) - O\left(\frac{x(\log n)^{1/3}}{n^{2/3}} + \frac{(\log n)^{1/3}}{n^{1/6}}\right)$ and consequently

(17) $$\Phi(s_{n,x}) = \Phi(x) + O\left(\frac{x(\log n)^{1/3}}{n^{2/3}} + \frac{(\log n)^{1/3}}{n^{1/6}}\right).$$

By a similar argument, equation (17) holds for $x \leq 0$ too. We combine the results from (13), (14), (15), (16), and (17) to obtain

$$P_n\left(\frac{\log \mathbf{B} - \mu_n}{\sigma_n} \leq x\right) = \sum_{\vec{W}} P_n(\vec{W})\left(\Phi(x) + O\left(\frac{x(\log n)^{1/3}}{n^{2/3}} + \frac{(\log n)^{1/3}}{n^{1/6}}\right)\right) + O\left(\frac{1}{n^4}\right)$$

$$= \Phi(x) + O\left(\frac{|x|(\log n)^{1/3}}{n^{2/3}} + \frac{(\log n)^{1/3}}{n^{1/6}}\right).$$

Therefore, for any $c > 0$, there is a constant $k_c > 0$ such that, for all n,

$$\sup_{|x| \leq c\sqrt{n}}\left|P_n\left(\frac{\log \mathbf{B} - \mu_n}{\sigma_n} \leq x\right) - \Phi(x)\right| \leq \frac{k_c(\log n)^{1/3}}{n^{1/6}}.$$

To complete the proof of Theorem 1, all that remains is to consider the case when $|x| > c\sqrt{n}$. However, this case is a standard Chernoff-type calculation using the moment generating function. If $x > c\sqrt{n}$, then

$$\left|P_n\left(\frac{\log \mathbf{B} - \mu_n}{\sigma_n} \leq x\right) - \Phi(x)\right| = \left|P_n\left(\frac{\log \mathbf{B} - \mu_n}{\sigma_n} > x\right) - (1 - \Phi(x))\right|$$

$$\leq P_n\left(\frac{\log \mathbf{B} - \mu_n}{\sigma_n} > x\right) + (1 - \Phi(x)) = P_n\left(e^{\log \mathbf{B}} > e^{\mu_n + x\sigma_n}\right) + \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt$$

$$\leq M_n(1)e^{-\mu_n - x\sigma_n} + \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt.$$

The integral is negligible because $e^{-t^2/2} \leq e^{-c\sqrt{n}t/2}$ for all $t > c\sqrt{n}$. As a crude but simple bound on $M_n(1) = E_n(\mathbf{B})$, we use the well-known fact that the maximum value of $\mathbf{B}$ over all compositions of n is no greater than $3^{n/3}$. If we choose c sufficiently large, then $3^{n/3}e^{-\mu_n - x\sigma_n}$ is also exponentially small. A similar argument applies when $x < 0$.

7. ADDITIONAL COMMENTS

A possible alternative approach to our problem is to use Hwang's quasi-powers theorem or related techniques ([**24**], page 645 of [**11**]). A second remark is that the referee's method of moments idea (outlined at the end of Section 3) could be used to prove asymptotic normality for more general sets S. However, we prefer to record the combinatorial technique of this paper in a relatively simple setting where technical details do not obscure the main ideas.

We thank the Perline brothers, Richard and Ron, for motivation [**29**] and helpful comments.

REFERENCES

1. M. ARCHIBALD, A. KNOPFMACHER: *The largest missing value in a composition of an integer.* Discrete Math., **311** (8–9) (2011), 723–731.

2. C. BANDERIER, P. HITCZENKO: *Enumeration and asymptotics of restricted compositions having the same number of parts.* Discrete Appl. Math. Preprint available at http://www.math.drexel.edu/~phitczen/BaHi_arx.pdf (2011).

3. E. A. BENDER, E. R. CANFIELD: *Locally restricted compositions I. Restricted adjacent differences.* Electron. J. Combin., **12** (2005), Research Paper 57, 27 pp. (electronic).

4. S. BERNSTEIN: *Sur l'extension du théoréme limite du calcul des probabilités aux sommes de quantités dépendantes.* Math. Ann., **97** (1) (1927), 1–59.

5. M. BÓNA, A. KNOPFMACHER: *On the probability that certain compositions have the same number of parts.* Ann. Comb., **14** (3) (2010), 291–306.

6. P. CHINN, S. HEUBACH: $(1,k)$-*compositions.* In: Proceedings of the Thirty-Fourth Southeastern International Conference on Combinatorics, Graph Theory and Computing (2003), vol. **164**, 183–194.

7. P. CHINN, S. HEUBACH: *Compositions of n with no occurrence of k.* In: Proceedings of the Thirty-Fourth Southeastern International Conference on Combinatorics, Graph Theory and Computing (2003), vol. **164**, 33–51.

8. P. CHINN, S. HEUBACH: *Integer sequences related to compositions without 2's.* J. Integer Seq., **6** (2) (2003), Article 03.2.3, 12 pp. (electronic).

9. W. FELLER: *An Introduction to Probability Theory and its Applications.* Vol. II. Second edition. John Wiley & Sons Inc., New York, 1971.

10. P. FLAJOLET, A. ODLYZKO: *Singularity analysis of generating functions.* SIAM J. Discrete Math., **3** (2) (1990), 216–240.

11. P. FLAJOLET, R. SEDGEWICK: *Analytic Combinatorics.* Cambridge University Press, Cambridge, 2009.

12. P. FLAJOLET, M. SORIA: *General combinatorial schemas*: *Gaussian limit distributions and exponential tails.* Discrete Math., **114** (1993), 159–180.

13. A. GUT, S. JANSON: *The limiting behaviour of certain stopped sums and some applications.* Scand. J. Stat., **10** (4) (1983), 281–292.

14. S. HEUBACH, T. MANSOUR: *Compositions of n with parts in a set.* In: Proceedings of the Thirty-Fifth Southeastern International Conference on Combinatorics, Graph Theory and Computing (2004), vol. **168**, 127–143.

15. S. HEUBACH, T. MANSOUR: *Combinatorics of Compositions and Words*, CRC Press, Boca Raton, 2009.

16. P. HITCZENKO, J. R. JOHNSON, H. J. HUANG: *Distribution of a class of divide and conquer recurrences arising from the computation of the Walsh-Hadamard transform.* Theoret. Comput. Sci., **352** (1–3) (2006), 8–30.

17. P. HITCZENKO, A. KNOPFMACHER: *Gap-free compositions and gap-free samples of geometric random variables.* Discrete Math., **294** (3) (2005), 225–239.

18. P. HITCZENKO, G. LOUCHARD: *Distinctness of compositions of an integer: a probabilistic analysis.* Random Structures Algorithms, **19** (3–4) (2001), 407–437. Analysis of algorithms (Krynica Morska, 2000).

19. P. HITCZENKO, C. D. SAVAGE: *On the probability that a randomly chosen part size in a random composition is unrepeated.* In: Paul Erdős and his mathematics (Budapest, 1999). János Bolyai Math. Soc., Budapest, (1999), 108–111.

20. P. HITCZENKO, C. D. SAVAGE: *On the multiplicity of parts in a random composition of a large integer.* SIAM J. Discrete Math., **18** (2) (2004), 418–435 (electronic).

21. P. HITCZENKO, G. STENGLE: *Expected number of distinct part sizes in a random integer composition.* Combin. Probab. Comput., **9** (6) (2000), 519–527.

22. H.-K. HWANG: *On convergence rates in the central limit theorems for combinatorial structures.* European J. Combin., **19** (3) (1998), 329–343.

23. H.-K. HWANG, S. JANSON: *A central limit theorem for random ordered factorizations of integers.* Electron. Commun. Probab., **16** (12) (2011), 347–361.

24. H.-K. HWANG, Y. N. YEH: *Measures of distinctness for random partitions and compositions of an integer.* Adv. in Appl. Math., **19** (3) (1997), 378–414.

25. A. KNOPFMACHER, H. PRODINGER: *On Carlitz compositions.* European J. Combin., **19** (5) (1998), 579–589.

26. A. KNOPFMACHER, B. RICHMOND: *Compositions with distinct parts.* Aequationes Math., **49** (1–2) (1995), 86–97.

27. M. E. MALANDRO: *Asymptotics for restricted integer compositions.* Preprint available at http://arxiv.org/pdf/1108.0337v1 (2011).

28. A. ODLYZKO, B. RICHMOND: *On the compositions of an integer.* In: Combinatorial mathematics, VII (Proc. Seventh Australian Conf., Univ. Newcastle, Newcastle, 1979), vol. **829** of Lecture Notes in Math. Springer, Berlin, (1980), 199–210.

29. R. PERLINE: *Zipf's law, the central limit theorem, and the random division of the unit interval.* Phys. Rev. E (3), **54** (1) (1995), 220–223.

30. A. RÉNYI: *On the asymptotic distribution of the sum of a random number of independent random variables.* Acta Math. Hungar., **8** (1957), 193–199.

31. C. Shapcott: *Part-products of* 1*-free integer compositions.* Electron. J. Comb., **18** (2011), Research Paper 235.

32. C. Shapcott: *Part-Products of Random Integer Compositions.* PhD thesis, Drexel University, 2012.

Department of Mathematics,
Drexel University,
Philadelphia, PA 19123
U.S.A.
E-mail: Eric.Jonathan.Schmutz@drexel.edu

Department of Mathematical Sciences,
Indiana University South Bend,
South Bend, IN 46634
U.S.A.
E-mail: shapcott@iusb.edu