

ON GROUPS ADMITTING NO INTEGRAL CAYLEY GRAPHS BESIDES COMPLETE MULTIPARTITE GRAPHS

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Let G be a non-trivial finite group, $S \subseteq G \setminus \{e\}$ be a set such that if $a \in S$, then $a^{-1} \in S$ and e be the identity element of G . Suppose that $\text{Cay}(G, S)$ is the Cayley graph with the vertex set G such that two vertices a and b are adjacent whenever $ab^{-1} \in S$. An arbitrary graph is called integral whenever all eigenvalues of the adjacency matrix are integers. We say that a group G is Cayley integral simple whenever every connected integral Cayley graph on G is isomorphic to a complete multipartite graph. In this paper we prove that if G is a non-simple group, then G is Cayley integral simple if and only if $G \cong \mathbb{Z}_{p^2}$ for some prime number p or $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Moreover, we show that there exist finite non-abelian simple groups which are not Cayley integral simple.

1. INTRODUCTION

A graph is called integral whenever all eigenvalues of the adjacency matrix are integers. In 1974, HARARY and SCHWENK have first introduced the notion of an integral graph [8]. The characterization of integral graphs seems very difficult so that it is better to concentrate on some special types of graphs. Let G be a finite non-trivial group and S be a subset of $G \setminus \{e\}$ such that $S = S^{-1}$, where e is the identity element of G . The Cayley graph $\text{Cay}(G, S)$ is the graph, whose vertex set is G and two vertices $a, b \in G$ are adjacent whenever $ab^{-1} \in S$. Recall that $\text{Cay}(G, S)$ is a k -regular graph, where $k = |S|$, and $G = \langle S \rangle$ if and only if $\text{Cay}(G, S)$

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is connected. If $\text{Cay}(G, S)$ is disconnected, then each connected component of $\text{Cay}(G, S)$ is isomorphic to $\text{Cay}(\langle S \rangle, S)$. Therefore, since we are considering only integral graphs and obviously a graph is integral if and only if each its connected component is such, we only study connected Cayley graphs.

Let us give a short review on literature about integral Cayley graphs. Every Cayley graph over a cyclic group is circulant and integral circulant graphs are characterized by WASIN SO (see Theorem 7.1 of [14], see Theorem 2.5 below). Unitary Cayley graphs are a family of integral circulant graphs which are denoted by $X_n = \text{Cay}(\mathbb{Z}_n, U_n)$, where U_n is the set of units of \mathbb{Z}_n (see [10]). The graphs X_n are Kronecker products of complete multipartite graphs except for X_{p^n} , where p is prime, which is complete multipartite (see [13]). WALTER KLOTZ and TORSTEN SANDER showed in [11] that if S belongs to the Boolean algebra generated by the subgroups of an abelian group G , then $\text{Cay}(G, S \setminus \{e\})$ is integral; recall that the Boolean algebra generated by a family of subsets of a set is obtained by arbitrary finite intersections, unions and complements of the subsets in the family. Later ROGER C. ALPERIN and BRIAN L. PETERSON showed in [4] that the Boolean algebra generated by the subgroups of an abelian group G is equal to the Boolean algebra generated by the integral sets of G , where a subset A of a group G is called integral whenever $\sum_{a \in A} \chi(a)$ is integer for every irreducible character χ of G . They

also proved that every atom of the Boolean algebra of subgroups of a finite group G is $\{b \in G \mid \langle a \rangle = \langle b \rangle\}$ for some $a \in G$, where the minimal non-empty subsets of a Boolean algebra are called atoms and it is well-known that every element of a Boolean algebra is expressible as a union of atoms. It follows that every Cayley graph $\text{Cay}(G, S)$ over an abelian group G is integral if and only if S is a finite unions of some atoms. Moreover, it is shown in [4] that the Boolean algebra generated by the subgroups of an arbitrary finite group is contained in the Boolean algebra generated by the integral sets of the group. This means that if S is a finite union of some atoms of the Boolean algebra of subgroups, then it is an integral set. Furthermore, if G is an abelian group, then by Theorem 9.8 of [9] and Theorem 3.1 of [6] due to LÁSZLÓ BABAI, $\text{Cay}(G, S)$ is integral but if G is a non-abelian group, then G has at least one irreducible character of degree greater than one and it is not clear how one must prove that the eigenvalues corresponding to the latter mentioned character are integer; it is shown in Section 4 that there exists a family of non-abelian simple groups G which can be generated by two elements a and b of orders 2 and 3, respectively and $\text{Cay}(G, (\langle a \rangle \cup \langle b \rangle) \setminus \{e\})$ is not integral.

It is evident that the Cayley graph $\text{Cay}(G, G \setminus \{e\})$ for any non-trivial finite group G is isomorphic to the complete graph of size $|G|$ and therefore it is an integral graph. The goal of the present paper is to find groups G admitting subsets S that lead to “non-trivial” (hence interesting) integral Cayley graphs. Cayley integral simple groups have been introduced in [2]: A finite group is called Cayley integral simple whenever complete graph is the only integral connected Cayley graph on the group. The Question 2.21 of [2] is then asked:

Question 1.1. *Is any finite simple group Cayley integral simple?*

The answer to this question is negative as Proposition 2.6 shows that for the complement $G \setminus H$ of any proper subgroup H in a group G , the Cayley graph $\text{Cay}(G, G \setminus H)$ is a complete multipartite graph in which each part has the same size $|H|$ and so it is an integral graph. Question 1.1 has motivated us to “correct” the notion of “simple” (or “trivial”) for a connected Cayley graph to be integral. We call a finite group G a Cayley Integral Simple group (or for short a CIS-group) whenever the only integral connected Cayley graphs of G are complete multipartite. Hence the following question is naturally posed.

Question 1.2. *For which finite non-trivial groups, there exists a “non-trivial” connected integral Cayley graph?*

In Section 3 we prove the following result.

Theorem 1.3. *Let G be a finite non-simple group. Then G is a CIS-group if and only if $G \cong \mathbb{Z}_{p^2}$ for some prime number p or $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Having proved Theorem 1.3, to give a complete answer to Question 1.2 one needs to consider the case of non-abelian simple groups. In Section 4, we give an infinite family of non-abelian simple groups which are not CIS-groups. We end the introduction with the following question.

Question 1.4. *Which finite non-abelian simple groups are CIS-groups?*

2. PRELIMINARIES

In this section we state some facts which we need in the other sections. Suppose that Γ is a circulant graph. Consider $A[\Gamma] = a_{ij}$, the adjacency matrix of Γ , with the labeling $\{0, 1, \dots, n-1\}$. Then the following set is called a symbol or circulant set of Γ .

$$S(\Gamma) = \{k : a_{0,k} = 1\} \subseteq \{1, 2, \dots, n-1\}.$$

Theorem 2.5 (Theorem 7.1 in [14]). *Let Γ be a circulant graph on n vertices with symbol $S(\Gamma)$ and $B(d, n) = \{k : \gcd(k, n) = d, d < n\}$. Then Γ is integral if and only if $S(\Gamma)$ is a union of the $B(d, n)$'s.*

The following proposition is somewhat related to Proposition 2.2 of [3] and shows that almost all groups have complete multipartite graphs as integral Cayley graphs and every complete multipartite integral Cayley graph on a group has particular inverse closed set which is the complement of a subgroup of the group.

Proposition 2.6. *Let G be a finite group and $\text{Cay}(G, S)$ be a Cayley graph. Then $G \setminus S$ is a subgroup of G if and only if $\text{Cay}(G, S)$ is a complete multipartite graph. In particular a complete multipartite Cayley graph has equal number of vertices in each partition and the graph is integral.*

Proof. Suppose that $H = G \setminus S$ is a subgroup of G . Then we define a relation as following: $a \sim b$ if and only if $ab^{-1} \in H$. The relation is an equivalence relation. Thus G is union of $|G|/|H| = k$ distinct partitions as following:

$$\{H, Ha_1, \dots, Ha_{k-1}\}.$$

It is trivial to see that if two elements a and b are in one partition, then $ab^{-1} \in H$. Thus $ab^{-1} \notin S$ and a is not adjacent to b . On the other hand, if a and b belong to two distinct partitions, then $ab^{-1} \notin H$. Therefore $ab^{-1} \in S$ and a is adjacent to b . It follows that $\text{Cay}(G, S)$ is a complete multipartite graph whose number of vertices in each part is equal to the order of H .

Conversely, suppose that $\text{Cay}(G, S)$ is a complete multipartite graph. Then each partition of $\text{Cay}(G, S)$ has equal number of elements since the graph is regular. Let $a, b \in G \setminus S$ and $a \neq b$. Then $a, b \notin S$ and the identity element of G is not adjacent to a and b since $ae^{-1} = a \notin S$, $be^{-1} = b \notin S$. Therefore a, e belong to one partition of $\text{Cay}(G, S)$ and similarly b, e belong to one partition of $\text{Cay}(G, S)$. Thus a and b belong to one partition. It means that $ab^{-1} \notin S$ and $G \setminus S$ is a subgroup of G . \square

Let Γ_1 and Γ_2 be two graphs with vertex sets V_1 and V_2 , respectively. Then the cartesian product of the two graphs is Γ with vertex set $V_1 \times V_2$ such that two vertices (v_1, w_1) and (v_2, w_2) are adjacent whenever $v_1 = v_2$ and w_1 is adjacent to w_2 in Γ_2 or $w_1 = w_2$ and v_1 is adjacent to v_2 in Γ_1 . It is well-known that the cartesian product of two integral graphs is integral. Let A and B be the adjacency matrices of Γ_1 and Γ_2 , respectively. Then the adjacency matrix of Γ is $A \otimes I_s + I_r \otimes B$, where r and s are the numbers of vertices of Γ_1 and Γ_2 , respectively, and I_r is the $r \times r$ identity matrix and the operation \otimes is the tensor product of two matrices. Recall that if C and D are two matrices, then the eigenvalues of $C \otimes D$ are $\lambda\mu$, where λ and μ are the eigenvalues of C and D , respectively (see [7] and [15]).

3. THE MAIN THEOREM

In this section we prove the main theorem. Recall that a CIS-group is a group which every connected integral Cayley graph on the group is complete multipartite. At first two families of CIS-groups are introduced in the following lemma.

Lemma 3.7. *If $G \cong \mathbb{Z}_p$ or $G \cong \mathbb{Z}_{p^2}$ for some prime number p , then G is a CIS-group.*

Proof. It is well-known that every Cayley graph on a cyclic group has circulant adjacency matrix. Thus by Theorem 2.5 if $G \cong \mathbb{Z}_p$, then G is Cayley integral simple. Now, suppose that $G \cong \mathbb{Z}_{p^2}$. Then by Theorem 2.5, \mathbb{Z}_{p^2} has at most 4 integral Cayley graphs $\text{Cay}(\mathbb{Z}_{p^2}, S_1)$, $\text{Cay}(\mathbb{Z}_{p^2}, S_2)$, $\text{Cay}(\mathbb{Z}_{p^2}, S_3)$ and $\text{Cay}(\mathbb{Z}_{p^2}, S_4)$, where $S_1 = \emptyset$, $S_2 = B(1, p^2)$, $S_3 = B(p, p^2)$ and $S_4 = B(1, p^2) \cup B(p, p^2)$. It is obvious that integral connected Cayley graphs of \mathbb{Z}_{p^2} are $\text{Cay}(\mathbb{Z}_{p^2}, S_2)$ and $\text{Cay}(\mathbb{Z}_{p^2}, S_4)$ but $\mathbb{Z}_{p^2} \setminus S_2$ and $\mathbb{Z}_{p^2} \setminus S_4$ are subgroups of \mathbb{Z}_{p^2} . Therefore G is a CIS-group by Proposition 2.6.

Lemma 3.8. *If $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then G is a CIS-group.*

Proof. It is trivial to see that the following graphs are all integral connected Cayley graphs of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

$$\begin{aligned} & \text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_2, \{(1, 0), (0, 1), (1, 1)\}), & \text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_2, \{(1, 0), (0, 1)\}) \\ & \text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_2, \{(1, 0), (1, 1)\}), & \text{Cay}(\mathbb{Z}_2 \times \mathbb{Z}_2, \{(1, 1), (0, 1)\}). \end{aligned}$$

But $\mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(1, 0), (0, 1), (1, 1)\}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(1, 0), (0, 1)\}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(1, 0), (1, 1)\}$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(1, 1), (0, 1)\}$ are subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore G is a CIS-group by Proposition 2.6. \square

It follows from the following theorem that the complement of an integral Cayley graph $\text{Cay}(G, S)$, i.e. $\text{Cay}(G, (G \setminus S) \setminus \{e\})$ is also integral.

Theorem 3.9 (Theorem 8.1 of chapter 1 in [15]). *Let Γ be a k -regular graph with n vertices. suppose that A and \bar{A} are the adjacency matrices of Γ and Γ^c , respectively. If the eigenvalues of A are $k = \lambda_1, \lambda_2, \dots, \lambda_n$, then the eigenvalues of \bar{A} are $n - 1 - \lambda_1, -1 - \lambda_2, \dots, -1 - \lambda_n$.*

Now, we can prove the main theorem. Suppose that J_m is the all one $m \times m$ matrix and I_m is the $m \times m$ identity matrix.

Proof of Theorem 1.3. We divide the proof into 5 cases. In all cases, we construct integral connected Cayley graphs which are not complete multipartite. In cases 1 and 2, abelian groups which are not simple are considered and in the other cases non-abelian non-simple groups are treated. In cases 3 to 5, where by the previous cases it is assumed G is non-abelian (the reader may note that in the case 4, the “non-abelian” hypothesis on the group is not used), we use the non-simplicity hypothesis on G , i.e. the existence of a non-trivial proper normal subgroup H to construct non-trivial connected integral Cayley graphs. In the case 3, groups G such that the factor group G/H is cyclic and the existence of an element $a \in G \setminus H$ of the order $\left| \frac{G}{H} \right|$ are discussed; and in the case 4, groups G for which the factor group G/H is not cyclic of prime order are ruled out; and finally in the case 5, the remaining case, i.e. groups G with G/H of prime order are dealt with.

Case 1. $G \cong \mathbb{Z}_{p^n}$, $n > 2$.

Set $S = B(1, p^n) \cup B(p^2, p^n)$. By Theorem 2.5, $\text{Cay}(\mathbb{Z}_{p^n}, S)$ is integral connected since every Cayley graph on cyclic group has circulant adjacency matrix and $\langle S \rangle = \mathbb{Z}_{p^n}$. On the other hand, $\mathbb{Z}_{p^n} \setminus S$ is not a subgroup of \mathbb{Z}_{p^n} since $p^2 + p \notin S$ and $p \notin S$ but $(p^2 + p) - p = p^2 \in S$. Thus \mathbb{Z}_{p^n} for $n > 2$ is not a CIS-group and therefore G is not a CIS-group.

Case 2. $G \cong \mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \dots \times \mathbb{Z}_{p_r^{n_r}}$, $r \geq 2$ and $G \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Consider

$$S = \underbrace{B(1, p_1^{n_1}) \times \{0\} \times \{0\} \times \dots \times \{0\}}_r$$

$$\cup \underbrace{\{0\} \times B(1, p_2^{n_2}) \times \{0\} \times \cdots \times \{0\}}_r \cup \cdots \cup \underbrace{\{0\} \times \{0\} \times \{0\} \times \cdots \times B(1, p_r^{n_r})}_r.$$

By Theorem 2.5, it is easy to see that $\text{Cay}(G, S)$ is integral. On the other hand, $G \setminus S$ is not a subgroup of G since if $a = (1, 2, 0, 0, \dots, 0)$, $b = (1, 1, 0, 0, \dots, 0)$ and G is not an elementary abelian 2-group, then $a, b \in G \setminus S$ but $a - b = (0, 1, 0, 0, \dots, 0) \in S$. Additionally, if G is an elementary abelian 2-group and $a = (1, 1, 1, 0, \dots, 0)$, $b = (1, 1, 0, 0, \dots, 0)$, then $a, b \in G \setminus S$ but $a - b = (0, 0, 1, 0, \dots, 0) \in S$. Furthermore, it is not hard to see that $G = \langle S \rangle$ and so $\text{Cay}(G, S)$ is connected. Therefore G is not a CIS-group.

Case 3. The group G is a non-abelian finite group and H is a proper normal subgroup of G such that $\frac{G}{H}$ is cyclic of arbitrary order k and there exists $a \in G \setminus H$ of order k .

It is obvious that,

$$G = H \cup aH \cup \cdots \cup a^{k-1}H.$$

Set $S = (H \setminus \{e\}) \cup \{a, a^2, \dots, a^{k-1}\}$, $|H| = m$. It is evident that $\langle S \rangle = G$. Moreover, $S = S^{-1}$ since the order of a is exactly k . On the other hand, if $|H| = 2$, then $\langle a \rangle \triangleleft G$ and G is isomorphic to $H \times \langle a \rangle$. But this is a contradiction since G is non-abelian. Therefore $G \setminus S$ is not a subgroup of G since if h_1 and h_2 are two distinct and nonidentity elements of H , then $ah_1, ah_2 \in G \setminus S$ but $(ah_1)(ah_2)^{-1} \notin G \setminus S$. Consider A which is the adjacency matrix of $\text{Cay}(G, S)$ with the labeling $\{H, aH, \dots, a^{k-1}H\}$ such that $H = \{h_1, h_2, \dots, h_m\}$ is fixed by the indices.

$$A = \begin{bmatrix} J_m - I_m & I_m & \cdots & I_m & I_m \\ I_m & J_m - I_m & & I_m & I_m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_m & I_m & \cdots & J_m - I_m & I_m \\ I_m & I_m & \cdots & I_m & J_m - I_m \end{bmatrix}$$

The adjacency matrix A has k^2 blocks, where the diagonal blocks are $J_m - I_m$ and the other blocks are all I_m . It follows that $A = (J_k - I_k) \otimes I_m + I_k \otimes (J_m - I_m)$. Thus $\text{Cay}(G, S)$ is a cartesian product of two complete graphs with m and k vertices and it is integral. Therefore G is not a CIS-group.

Case 4. G is a finite group and H is a non-trivial proper normal subgroup of G such that $\frac{G}{H}$ is not cyclic of prime order.

There exists $a \in G \setminus H$ such that the order of aH is p , where p is prime and

$$G \neq H \cup aH \cup \cdots \cup a^{p-1}H.$$

Set $S = G \setminus (aH \cup \cdots \cup a^{p-1}H \cup \{e\})$, $[G : H] = m$ and $|H| = r$. Thus $pk = m$ for a natural number $k \geq 2$ since $p|m$ and $\frac{G}{H}$ is not cyclic of prime order. Furthermore, $S = S^{-1}$ since H is normal in G and $G = \langle S \rangle$ as the order of S is greater than $\frac{|G|}{2}$.

On the other hand, $G \setminus S$ is not a subgroup of G since if h is a nonidentity element of H , then $a, ah \in G \setminus S$ but $aha^{-1} \notin G \setminus S$. Consider the following labeling for G :

$$\{H, aH, \dots, a^{p-1}H, b_2H, ab_2H, \dots, a^{p-1}b_2H, \dots, b_kH, ab_kH, \dots, a^{p-1}b_kH\},$$

where $H = \{h_1, h_2, \dots, h_r\}$ is fixed by the indices and also b_2, \dots, b_k are distinct elements of $G \setminus H$ such that

$$\begin{aligned} b_2H &\notin \{a^iH \mid 0 \leq i \leq p-1\}, \\ b_3H &\notin \{a^i b_2H \mid 0 \leq i \leq p-1\} \cup \{a^iH \mid 0 \leq i \leq p-1\}, \dots, \\ b_kH &\notin \{a^i b_jH \mid 0 \leq i \leq p-1, 2 \leq j \leq k-1\} \cup \{a^iH \mid 0 \leq i \leq p-1\}, \end{aligned}$$

where a^0 is defined the identity element of G .

The following matrix is the adjacency matrix of $\text{Cay}(G, S)$ with the labeling.

$$A = \begin{bmatrix} J_r - I_r & 0 & \dots & 0 & & J_r & J_r & \dots & J_r \\ 0 & J_r - I_r & & 0 & & J_r & J_r & \dots & J_r \\ \vdots & & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & J_r - I_r & & J_r & J_r & \dots & J_r \\ \ddots & & \ddots & & \dots & & \ddots & & \ddots \\ J_r & J_r & \dots & J_r & & J_r - I_r & 0 & \dots & 0 \\ J_r & J_r & \dots & J_r & & 0 & J_r - I_r & & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots & & \ddots & \vdots \\ J_r & J_r & \dots & J_r & & 0 & \dots & 0 & J_r - I_r \end{bmatrix}$$

The adjacency matrix A has k^2 blocks, where the diagonal blocks are $I_p \otimes (J_r - I_r)$ and the other blocks are all $J_p \otimes J_r$. If we consider \bar{A} , the adjacency matrix of complement of the graph, then $\bar{A} = (I_k \otimes (J_p - I_p)) \otimes J_r$. Therefore $\text{Cay}(G, S)$ is integral and G is not a CIS-group.

Case 5. The group G is a finite non-abelian group and H is a proper normal subgroup of G such that $\frac{G}{H}$ is of prime order p . By the case 3 we may assume that all elements of order p belong to H . It follows that H is not of prime order.

Suppose that $\frac{G}{H} = \langle bH \rangle$ and $o(b) = m$. Then $b^p = h$ and $o(h) = \frac{m}{\gcd(p, m)} >$

1. Set $\frac{m}{\gcd(p, m)} = qi$ for some prime number q . It is evident that

$$G = H \cup bH \cup \dots \cup b^{p-1}H.$$

Set

$$S = bH \cup \dots \cup b^{p-1}H \cup \langle h_1 \rangle \setminus \{e\},$$

where $h_1 = h^i$. Furthermore, $S = S^{-1}$ since H is normal in G . If $\langle h_1 \rangle = \{e\}$, then $h^i = e$ but this is a contradiction as the order of h is qi . It implies that $\langle S \rangle = G$

since the order of S is greater than $\frac{|G|}{2}$. On the other hand, if $\langle h_1 \rangle = H$, then $|H| = q$ and it is not the case since the order of H is not prime. Therefore $G \setminus S$ is not a subgroup of G . Set $|H| = k$ and consider the adjacency matrix of $\text{Cay}(G, S)$ with the labeling $\{H, bH, \dots, b^{p-1}H\}$, where $H = \{h_1, h_2, \dots, h_k\}$ is fixed by the indices.

$$B = \begin{bmatrix} A & J_k & \dots & J_k & J_k \\ J_k & A & \dots & J_k & J_k \\ \vdots & & \ddots & \vdots & \vdots \\ J_k & J_k & \dots & A & J_k \\ J_k & J_k & \dots & J_k & A \end{bmatrix}$$

where A is the adjacency matrix of $\text{Cay}(H, (\langle h_1 \rangle \setminus \{e\}))$. If $x_1 x_2^{-1} \in \langle h_1 \rangle \setminus \{e\}$, then $x_1 x_2^{-1} = h_1^t$, $1 \leq t < q$ and it follows that for every natural number $0 \leq r \leq p-1$,

$$b^r x_1 x_2^{-1} b^{-r} = b^r h_1^t b^{-r} = b^r (b^{pi})^t b^{-r} = (b^{pi})^t = h_1^t = x_1 x_2^{-1}.$$

Therefore the diagonal of matrix B is A . If we consider the complement of $\text{Cay}(G, S)$, then it is obvious that it is integral and therefore $\text{Cay}(G, S)$ is integral. As a consequence, G is not a CIS-group.

4. FINITE NON-ABELIAN SIMPLE GROUPS WHICH ARE NOT CIS-GROUPS

In this section we prove that there exists an infinite family of finite non-abelian simple groups which are not CIS-groups. This means that if G is a simple group, then we can not conclude that it is a CIS-group in general.

Another point that should be explained is as follows: Given the background that in finite abelian groups the union of any two cyclic subgroups create integral Cayley graph, the reader may ask why one could not just pick two cyclic subgroups (maybe place some restrictions on their orders) and join their generating sets into one subset S . Wouldn't this most probably yield some interesting integral Cayley graphs in the case that G is simple? The answer in general is negative: Let G be any finite group of order greater than 24 which is generated by an involution a and an element b of order 3, then $\text{Cay}(G, (\langle a \rangle \cup \langle b \rangle) \setminus \{e\})$ is a connected 3-regular graph which is not integral. This is because the order of a connected integral cubic Cayley graph is at most 24 (see Theorem 1.1 of [1]). On the other hand, the number of such groups G with the mentioned generating set $\{a, b\}$ abound, e.g. the alternating group A_n for $n \geq 3$ has such a generating set whenever $n \notin \{3, 6, 7, 8\}$ (see [12]).

The main result of this section is the following.

Theorem 4.10. *Let G be a finite group and H, K be its proper subgroups such that $HK = G$ and $H \cap K = \{e\}$. If there exists an integral Cayley graph $\text{Cay}(H, S)$ such that $S \cup \{e\}$ is not a subgroup of H , then G is not a CIS-group.*

Proof. Suppose that $\text{Cay}(H, S)$ is integral. Then we claim that $\text{Cay}(G, S)$ is integral. Define

$$\varphi : \text{Cay}(H, S) \times \text{Cay}(K, \emptyset) \rightarrow \text{Cay}(G, S) \quad \text{and} \quad (h, k)\varphi = hk,$$

where the multiplication of the left side of the map is cartesian product of two graphs. It is trivial to see that φ is surjective since $HK = G$ and therefore it is bijective since the order of $\text{Cay}(H, S) \times \text{Cay}(K, \emptyset)$ and $\text{Cay}(G, S)$ are equal as $G = HK$ and $H \cap K = \{e\}$. Suppose that (h_1, k_1) is adjacent to (h_2, k_2) . Then $k_1 = k_2$ and $h_1 h_2^{-1} \in S$. It follows that $h_1 k_1 k_2^{-1} h_2^{-1} \in S$ and therefore $h_1 k_1$ is adjacent to $h_2 k_2$. On the other hand, if $h_1 k_1$ is adjacent to $h_2 k_2$, then (h_1, k_1) is adjacent to (h_2, k_2) since the two graphs are regular of the same degree and if (h_1, k_1) is adjacent to (h_2, k_2) , then $h_1 k_1$ is adjacent to $h_2 k_2$. Thus $\text{Cay}(H, S) \times \text{Cay}(K, \emptyset) \cong \text{Cay}(G, S)$ and furthermore $\text{Cay}(G, S)$ is integral. But the complement of $\text{Cay}(G, S)$, i.e. $\text{Cay}(G, (G \setminus S) \setminus \{e\})$ is integral. Additionally, it is connected since $\text{Cay}(G, S)$ is disconnected. On the other hand, $S \cup \{e\}$ is not a subgroup of G as it is not a subgroup of H . Therefore G is not a CIS-group.

REMARK 4.11. One must note that proper subgroups H and K with the given property:

$$G = HK \quad \text{and} \quad H \cap K = \{e\}$$

are not necessarily unique; e.g., the symmetric group on degree 3, one has

$$S_3 = \langle (1, 2) \rangle \langle (1, 2, 3) \rangle = \langle (1, 3) \rangle \langle (1, 2, 3) \rangle = \langle (2, 3) \rangle \langle (1, 2, 3) \rangle.$$

The topic of “products of groups” is widely studied. The reader may look for some further details the book [5].

The following corollary shows that there exists a family of finite non-abelian simple groups which are not CIS-groups since the alternating group A_n , $n \geq 5$ is simple.

Corollary 4.12. *The alternating group A_n is not a CIS-group, where $n \geq 5$ is prime.*

Proof. Suppose that $n \geq 5$ is a prime number. Then $A_n = A_{n-1}K$, where $K = \langle (1, 2, \dots, n) \rangle$ and $A_{n-1} \cap K = \{e\}$. If $S = A_{n-1} \setminus A_{n-2}$, then $\text{Cay}(A_{n-1}, S)$ is integral and $S \cup \{e\}$ is not a subgroup of A_{n-1} . Therefore by the previous theorem A_n is not a CIS-group.

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