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AN ALTERNATE CIRCULAR SUMMATION FORMULA OF THETA FUNCTIONS AND ITS APPLICATIONS

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We prove a general alternate circular summation formula of theta functions, which yields a great deal of theta-function identities. In particular, we recover several identities in Ramanujan's Notebook from this identity. We also obtain two formulas for $(q;q)_\infty^{2n}$.

1. INTRODUCTION AND MAIN RESULTS

Throughout the paper we put $q = e^{2\pi i\tau}$, where Im $\tau > 0$. The Jacobi theta functions $\theta_k(z|\tau)$ for $k = 1, 2, 3, 4$ are defined as follows:

$$\theta_1(z|\tau) = -iq^{\frac{1}{8}} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(n+1)}{2}} e^{(2n+1)iz}, \quad \theta_2(z|\tau) = q^{\frac{1}{8}} \sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}} e^{(2n+1)iz},$$

$$\theta_3(z|\tau) = \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} e^{2niz}, \qquad \theta_4(z|\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n^2}{2}} e^{2niz}.$$

To carry out our work, we need some notations and basic facts about the Jacobi theta functions. We use the familiar notation

$$(z;q)_\infty = \prod_{n=0}^{\infty} (1 - zq^n)$$

and also

$$(a, b, \cdots, c; q)_\infty = (a;q)_\infty (b;q)_\infty \cdots (c;q)_\infty.$$

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Using the well-known Jacobi product identity [**2**, p. 35, Entry 19]

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = (ab, -a, -b; ab)_\infty, \tag{1}$$

we can deduce the infinite product representations for theta functions, namely,

$$\begin{aligned}
\theta_1(z|\tau) &= iq^{\frac{1}{8}} e^{-iz}(q, e^{2iz}, qe^{-2iz}; q)_\infty,\\
\theta_2(z|\tau) &= q^{\frac{1}{8}} e^{-iz}(q, -e^{2iz}, -qe^{-2iz}; q)_\infty,\\
\theta_3(z|\tau) &= \Big(q, -q^{\frac{1}{2}} e^{2iz}, -q^{\frac{1}{2}} e^{-2iz}; q\Big)_\infty,\\
\theta_4(z|\tau) &= \Big(q, q^{\frac{1}{2}} e^{2iz}, q^{\frac{1}{2}} e^{-2iz}; q\Big)_\infty.
\end{aligned} \tag{2}$$

Employing the above identities, we easily get the relations:

$$\begin{aligned}
&\theta_1(z+\pi|\tau) = -\theta_1(z|\tau), \qquad \theta_2(z+\pi|\tau) = -\theta_2(z|\tau),\\
&\theta_3(z+\pi|\tau) = \theta_3(z|\tau), \qquad \theta_4(z+\pi|\tau) = \theta_4(z|\tau)
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
&\theta_1(z+\pi\tau|\tau) = -q^{-\frac{1}{2}} e^{-2iz}\theta_1(z|\tau), \quad \theta_2(z+\pi\tau|\tau) = q^{-\frac{1}{2}} e^{-2iz}\theta_2(z|\tau),\\
&\theta_3(z+\pi\tau|\tau) = q^{-\frac{1}{2}} e^{-2iz}\theta_3(z|\tau), \quad \theta_4(z+\pi\tau|\tau) = -q^{-\frac{1}{2}} e^{-2iz}\theta_4(z|\tau).
\end{aligned} \tag{4}$$

We also have

$$\begin{aligned}
&\theta_1\Big(z+\frac{\pi}{2}|\tau\Big) = \theta_2(z|\tau), \qquad \theta_2\Big(z+\frac{\pi}{2}|\tau\Big) = -\theta_1(z|\tau),\\
&\theta_3\Big(z+\frac{\pi}{2}|\tau\Big) = \theta_4(z|\tau), \qquad \theta_4\Big(z+\frac{\pi}{2}|\tau\Big) = \theta_3(z|\tau)
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
&\theta_1(z+\frac{\pi\tau}{2}|\tau) = iq^{-\frac{1}{8}} e^{-iz}\theta_4(z|\tau), \quad \theta_2\Big(z+\frac{\pi\tau}{2}|\tau\Big) = q^{-\frac{1}{8}} e^{-iz}\theta_3(z|\tau),\\
&\theta_3\Big(z+\frac{\pi\tau}{2}|\tau\Big) = q^{-\frac{1}{8}} e^{-iz}\theta_2(z|\tau), \quad \theta_4\Big(z+\frac{\pi\tau}{2}|\tau\Big) = iq^{-\frac{1}{8}} e^{-iz}\theta_1(z|\tau).
\end{aligned} \tag{6}$$

We also need the following special case of $f(a,b)$:

$$\varphi(q) := f(q,q), \quad \psi(q) := f(q,q^3). \tag{7}$$

Definitions (7) can be found in [**2**, p. 36–37, Entry 22]. The following properties of $\varphi(q)$ and $\psi(q)$ can be verified through simple computations.

$$\varphi(q) = \theta_3(0|2\tau) = (q^2, -q, -q; q^2)_\infty, \tag{8}$$

$$\varphi(-q) = \theta_4(0|2\tau) = (q^2, q, q; q^2)_\infty = (q;q)_\infty(q;q^2)_\infty = \frac{(q;q)_\infty}{(-q;q)_\infty}, \tag{9}$$

(10) $\psi(q) = \theta_2(\pi\tau|4\tau) = \theta_3(\pi\tau|4\tau) = \dfrac{q^{-\frac{1}{8}}}{2}\theta_2(0|\tau) = (q,-q,-q;q)_\infty = \dfrac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$

(11) $\psi(-q) = -i\theta_1(\pi\tau|4\tau) = \theta_4(\pi\tau|4\tau) = (q;q)_\infty(-q^2;q^2)_\infty = \dfrac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}.$

On page 54 of his Lost Notebook [**12**], Ramanujan recorded the following statement (translated here in terms of $\theta_3(z|\tau)$).

Theorem 1.1. *For any positive integer $n \geq 2$,*

$$\sum_{k=0}^{n-1} q^{k^2} e^{2kiz}\theta_3^n(z+k\pi\tau|n\tau) = \theta_3(z|\tau)F_n(\tau). \tag{12}$$

When $n \geq 3$,

$$F_n(\tau) = 1 + 2nq^{n-1} + \cdots . \tag{13}$$

The proof of (12) was first given by Rangachari [**13**], and later, Son [**15**]. Several authors have obtained results related to evaluation of $F_n(\tau)$ for the integers n not found in Ramanujan's work (see [**1, 6, 7, 11**]). The first proof of the entire Theorem 1.1 was given by H. H. Chan, Z.-G. Liu and S. T. Ng [**4**], i.e., (12) and (13). By applying the Jacobi imaginary transformation to (12), Chan, Liu and Ng [**4**] got

Theorem 1.2. *For any positive integer n, there exists a quantity $G_n(\tau)$ such that*

$$\sum_{k=0}^{n-1} \theta_3^n\left(z + \frac{k\pi}{n}\middle|\tau\right) = G_n(\tau)\theta_3(nz|n\tau).$$

M. Boon et al. [**3**] proved the following additive decomposition of $\theta_3(z|\tau)$ (see [**3**, Eq. (7)]).

Theorem 1.3. *For any positive integer n, we have*

$$\sum_{k=0}^{n-1} \theta_3(z+k\pi|\tau) = n\theta_3(nz|n^2\tau).$$

Inspired by [**4**] and [**3**], X.-F. Zeng [**17**] proved the following important formula unifying Theorem 1.2 and 1.3 .

Theorem 1.4. *For any positive integer m, n, a and b with $a+b=n$, there exists a quantity $G_{a,b,m,n}(y|\tau)$ such that*

$$\sum_{k=0}^{mn-1} \theta_3^a\left(z + \frac{y}{a} + \frac{k\pi}{mn}|\tau\right)\theta_3^b\left(z - \frac{y}{a} + \frac{k\pi}{mn}|\tau\right) = G_{a,b,m,n}(y|\tau)\theta_3(mnz|m^2n\tau).$$

Recently, S. H. Chan and Z.-G. Liu [**5**] eliminated the unnecessary restrictions in Zeng's Theorem 1.4 and added many free parameters into it. S. H. Chan and Z.-G. Liu [**5**] extended Zeng's theorem to the following more general and concise form.

Theorem 1.5. *Suppose that $y_1, y_2, \ldots, y_n$ are complex numbers such that $y_1+y_2+\cdots+y_n=0$. Then there exists a quantity $G_{m,n}(y_1,y_2,\cdots,y_n|\tau)$ such that*

$$\sum_{k=0}^{mn-1}\prod_{j=1}^{n}\theta_3\left(z+y_j+\frac{k\pi}{mn}|\tau\right)=G_{m,n}(y_1,y_2,\cdots,y_n|\tau)\theta_3(mnz|m^2n\tau).$$

Boon et al. [**3**] also obtained the following alternate circular summation formula (see the second identity in [**3**, Eq. (8)].

Theorem 1.6. *For any positive integer n, we have*

$$\sum_{k=0}^{2m-1}(-1)^k\theta_3\left(z+\frac{k\pi}{2m}|\tau\right)=2m\theta_2(2mz|4m^2\tau). \tag{14}$$

Motivated by [**3**] and [**5**], we obtain the following general alternate circular summation formula of theta functions.

Theorem 1.7. *Suppose that m and n are such positive integers that mn is even and $y_1, y_2, \ldots, y_n$ are complex numbers such that $y_1+y_2+\cdots+y_n=0$. Then we have*

$$\sum_{k=0}^{mn-1}(-1)^k\prod_{j=1}^{n}\theta_3\left(z+y_j+\frac{k\pi}{mn}|\tau\right)=H_{m,n}(y_1,y_2,\ldots,y_n|\tau)\theta_2(mnz|m^2n\tau), \tag{15}$$

where

$$\begin{aligned} H_{m,n}(y_1,y_2,\ldots,y_n|\tau) &= mnq^{-\frac{m^2n}{8}} \\ &\quad\times\sum_{\substack{s_1,\ldots,s_n=-\infty \\ s_1+\cdots+s_n=\frac{mn}{2}}}^{\infty} q^{\frac{1}{2}(s_1^2+s_2^2+\cdots+s_n^2)}e^{2i(s_1y_1+s_2y_2+\cdots+s_ny_n)}. \end{aligned} \tag{16}$$

Note that if m and n are such positive integers that mn is odd, the summations on the left-hand side of (15) are not "circular". For convenience, in what follows we always denote $H_{m,n}(0,0,\ldots,0|\tau)$ simply as $H_{m,n}(\tau)$.

The following sections will be organized as follows. In Section 2, the proof of Theorem 1.7 will be given. In Section 3, we deduce Theorem 1.6 and the ensuing Corollaries 1.8 and 1.9 from Theorem 1.7.

Corollary 1.8. *We have*

$$\sum_{k=0}^{2n-1}(-1)^k\theta_3^{2n}\left(z+\frac{k\pi}{2n}|\tau\right)=H_{1,2n}(\tau)\theta_2(2nz|2n\tau). \tag{17}$$

Corollary 1.9. *We have*

$$\sum_{k=0}^{2m-1}(-1)^k\theta_3\left(z+y+\frac{k\pi}{2m}|\tau\right)\theta_3\left(z-y+\frac{k\pi}{2m}|\tau\right) \tag{18}$$
$$=\begin{cases}2m\theta_2(2y|2\tau)\theta_2(2mz|2m^2\tau), & \textit{if } m \textit{ is odd;}\\ 2m\theta_3(2y|2\tau)\theta_2(2mz|2m^2\tau), & \textit{if } m \textit{ is even.}\end{cases}$$

In Section 3, this identity will be discussed in detail. We obtain many modular identities from (18). In particular, several identities in Ramanujan's notebook will be recovered.

In Section 4, we give the proofs of the following two identities for $(q;q)_\infty^{2n}$, respectively, using Theorem 1.7.

Corollary 1.10. *We have*

$$(q;q)_\infty^{2n}=q^{-n}(q^{2n};q^{2n})_\infty \sum_{\substack{s_1,s_2,\ldots,s_{2n}=-\infty\\ s_1+s_2+\cdots+s_{2n}=2n}}^{\infty} q^{\frac{1}{2}(s_1^2+s_2^2+\cdots+s_{2n}^2)}e^{\frac{\pi i}{2n}\sum_{\ell=1}^{n}(s_\ell-s_{n+\ell})(2\ell-1)}. \tag{19}$$

Corollary 1.11. *We have*

$$(q^{2n};q^{2n})_\infty^{2n}=q^{-\frac{n}{2}}(q;q)_\infty \sum_{\substack{s_1,s_2,\ldots,s_{2n}=-\infty\\ s_1+s_2+\cdots+s_{2n}=n}}^{\infty} q^{n(s_1^2+s_2^2+\cdots+s_{2n}^2)+\frac{1}{4}\sum_{\ell=1}^{n}(s_\ell-s_{n+\ell})(2\ell-1)}. \tag{20}$$

2. PROOF OF THEOREM 1.7

Proof. For any positive integers m and n such that mn is even, we set

$$g(z)=\sum_{k=0}^{mn-1}(-1)^k\prod_{j=1}^{n}\theta_3\left(\frac{z}{mn}+y_j+\frac{k\pi}{mn}|\frac{\tau}{m^2n}\right).$$

Then we find that

$$\begin{aligned}g(z+\pi)&=\sum_{k=0}^{mn-1}(-1)^k\prod_{j=1}^{n}\theta_3\left(\frac{z}{mn}+y_j+\frac{(k+1)\pi}{mn}|\frac{\tau}{m^2n}\right)\\ &=\sum_{k=1}^{mn}(-1)^{k-1}\prod_{j=1}^{n}\theta_3\left(\frac{z}{mn}+y_j+\frac{k\pi}{mn}|\frac{\tau}{m^2n}\right)\\ &=\sum_{k=1}^{mn-1}(-1)^{k-1}\prod_{j=1}^{n}\theta_3\left(\frac{z}{mn}+y_j+\frac{k\pi}{mn}|\frac{\tau}{m^2n}\right)\end{aligned}$$

$$
\begin{aligned}
&-\prod_{j=1}^{n}\theta_3\left(\frac{z}{mn}+y_j+\pi|\frac{\tau}{m^2n}\right)\\
&=-\sum_{k=0}^{mn-1}(-1)^k\prod_{j=1}^{n}\theta_3\left(\frac{z}{mn}+y_j+\frac{k\pi}{mn}|\frac{\tau}{m^2n}\right)=-g(z).
\end{aligned}
$$

In the above deduction, we have used the condition that mn is even and the property of $\theta_3(z|\tau)$ in (3). Using the property of $\theta_3(z|\tau)$ in (4), we also have

$$
\begin{aligned}
g(z+\pi\tau)&=\sum_{k=0}^{mn-1}(-1)^k\prod_{j=1}^{n}\theta_3\left(\frac{z}{mn}+y_j+\frac{k\pi}{mn}+\frac{\pi\tau}{mn}|\frac{\tau}{m^2n}\right)\\
&=\sum_{k=0}^{mn-1}(-1)^k\prod_{j=1}^{n}\left[q^{-\frac{1}{2n}}e^{-2im\left(\frac{z}{mn}+y_j+\frac{k\pi}{mn}\right)}\theta_3\left(\frac{z}{mn}+y_j+\frac{k\pi}{mn}|\frac{\tau}{m^2n}\right)\right]\\
&=q^{-\frac{1}{2}}e^{-2iz}\sum_{k=0}^{mn-1}(-1)^k\prod_{j=1}^{n}\theta_3\left(\frac{z}{mn}+y_j+\frac{k\pi}{mn}|\frac{\tau}{m^2n}\right)=q^{-\frac{1}{2}}e^{-2iz}g(z).
\end{aligned}
$$

Therefore the function $F(z)=\dfrac{g(z)}{\theta_2(z|\tau)}$ is an elliptic function of z. It is well-known that $\theta_2(z|\tau)$ has only a simple zero at $z=\frac{\pi}{2}$ in the period parallelogram. This shows that $F(z)$ has at most one pole in its period parallelogram. Hence $F(z)$ is independent of z, say, it equals $S(y_1,y_2,\cdots,y_n|\tau)$. It follows that

$$
\sum_{k=0}^{mn-1}(-1)^k\prod_{j=1}^{n}\theta_3\left(\frac{z}{mn}+y_j+\frac{k\pi}{mn}|\frac{\tau}{m^2n}\right)=S(y_1,y_2,\ldots,y_n|\tau)\theta_2(z|\tau). \tag{21}
$$

Replacing z and τ in the above identity by mnz and $m^2n\tau$, respectively, gives (15).

Now it remains to determine $S(y_1,y_2,\cdots,y_n|\tau)$. To complete this, we compare the coefficients of e^{iz} on both sides of (21). Using the definition of $\theta_3(z|\tau)$, we easily get

$$
\begin{aligned}
&\prod_{j=1}^{n}\theta_3\left(\frac{z}{mn}+y_j+\frac{k\pi}{mn}|\frac{\tau}{m^2n}\right)\\
&=\sum_{s_1,\ldots,s_n=-\infty}^{\infty}q^{\frac{1}{2m^2n}(s_1^2+\cdots+s_n^2)}e^{\frac{2i(z+k\pi)}{mn}(s_1+\cdots+s_n)+2i(s_1y_1+\cdots+s_ny_n)}.
\end{aligned}
$$

Then the coefficient of e^{iz} of the above series is

$$
(-1)^k\sum_{\substack{s_1,\ldots,s_n=-\infty\\ s_1+\cdots+s_n=\frac{mn}{2}}}^{\infty}q^{\frac{1}{2m^2n}(s_1^2+s_2^2+\cdots+s_n^2)}e^{2i(s_1y_1+s_2y_2+\cdots+s_ny_n)}.
$$

The coefficient of e^{iz} on the left-hand side of (21) is

$$mn \sum_{\substack{s_1,\dots,s_n=-\infty \\ s_1+\cdots+s_n=\frac{mn}{2}}}^{\infty} q^{\frac{1}{2m^2n}(s_1^2+s_2^2+\cdots+s_n^2)} e^{2i(s_1y_1+s_2y_2+\cdots+s_ny_n)}.$$

From the definition of $\theta_2(z|\tau)$, we have

$$q^{\frac{1}{8}} S(y_1,y_2,\dots,y_n|\tau) = mn \sum_{\substack{s_1,\dots,s_n=-\infty \\ s_1+\cdots+s_n=\frac{mn}{2}}}^{\infty} q^{\frac{1}{2m^2n}(s_1^2+s_2^2+\cdots+s_n^2)} e^{2i(s_1y_1+s_2y_2+\cdots+s_ny_n)}.$$

This is (16), where $H(y_1,y_2,\dots,y_n|\tau) = S(y_1,y_2,\dots,y_n|m^2n\tau)$. This completes the proof. □

In the above proof, we have used the fact that an elliptic function with at most one pole in its period parallelogram is a constant (see, for example, [**16**, p. 432]).

Now, we split Theorem 1.7 into two cases according to the parity of m and n. Replacing m in Theorem 1.7 by $2m$ gives

Case 1 of Theorem 1.7. Suppose that m and n are any positive integers and $y_1, y_2, \cdots, y_n$ are complex numbers such that $y_1+y_2+\cdots+y_n=0$. Then we have

$$\begin{aligned}(22)\qquad &\sum_{k=0}^{2mn-1}(-1)^k \prod_{j=1}^{n} \theta_3\left(z+y_j+\frac{k\pi}{2mn}|\tau\right) \\ &= H_{2m,n}(y_1,y_2,\dots,y_n|\tau)\theta_2(2mnz|4m^2n\tau).\end{aligned}$$

We replace n in Theorem 1.7 by $2n$ to obtain

Case 2 of Theorem 1.7. Suppose that m and n are any positive integers and $y_1, y_2, \cdots, y_{2n}$ are complex numbers such that $y_1+y_2+\cdots+y_{2n}=0$. Then we have

$$\begin{aligned}(23)\qquad &\sum_{k=0}^{2mn-1}(-1)^k \prod_{j=1}^{2n} \theta_3\left(z+y_j+\frac{k\pi}{2mn}|\tau\right) \\ &= H_{m,2n}(y_1,y_2,\dots,y_{2n}|\tau)\theta_2(2mnz|2m^2n\tau).\end{aligned}$$

Note that both $H_{2m,n}(y_1,y_2,\dots,y_n|\tau)$ in (22) and $H_{m,2n}(y_1,y_2,\dots,y_{2n}|\tau)$ in (23) are defined by (16).

3. SOME SPECIAL CASES

Proof of Theorem 1.6. Putting $n=1$ in (22) gives Theorem 1.6. □

Proof of Corollary 1.8. Setting $m = 1$ and $y_j = 0$ in (23) gives (17). □

Replacing z in (17) by $z + \frac{\pi + \pi\tau}{2}$ and then using (5) and (6), we get

$$\sum_{k=0}^{2n-1} \theta_1^{2n}\left(z + \frac{k\pi}{2n}|\tau\right) = H_{1,2n}(\tau)\theta_3(2nz|2n\tau).$$

The left-hand side of this identity is the same as the even case of [**4**, Thm. 4.2]. We note that there is a misprint in [**4**, Thm. 4.2].

Proof of Corollary 1.9. From (16), we have

$$\begin{aligned}
H_{m,2}(y, -y|\tau) &= 2mq^{-\frac{m^2}{4}} \sum_{s=-\infty}^{\infty} q^{\frac{s^2+(m-s)^2}{2}} e^{2iy[s-(m-s)]} \\
&= 2mq^{\frac{m^2}{4}} e^{-2imy} \sum_{s=-\infty}^{\infty} q^{s^2} e^{2is(y-m\pi\tau)} \\
&= 2mq^{\frac{m^2}{4}} e^{-2imy}\theta_3(2y - m\pi\tau|2\tau) = \begin{cases} 2m\theta_2(2y|2\tau), & \text{if } m \text{ is odd;} \\ 2m\theta_3(2y|2\tau), & \text{if } m \text{ is even.} \end{cases}
\end{aligned}$$

Then setting $n = 1$ in (23), we obtain (18). □

Taking $m = 1$ in (18) and then using (5) gives

Proposition 3.12. *We have*

$$\theta_3(z + y|\tau)\theta_3(z - y|\tau) - \theta_4(z + y|\tau)\theta_4(z - y|\tau) = 2\theta_2(2y|2\tau)\theta_2(2z|2\tau).$$

This identity is equivalent to [**8**, Eq. (16)] and [**14**, Eq. (1.3d), p327]. J. A. Ewell [**9**, Eq. (1. 10)] deduced a sextuple product identity from this one. Z.-G. Liu and X.-M. Yang [**10**, Eq. (1.11) in Thm. 4] also deduced this identity from the Schröter formula. Using (5), (6) and the Jacobi imaginary transformation on this identity, all the other identities in [**10**, Thm. 4] can be deduced.

Setting $m = 2$ in (18), and then using (5), we get the following remarkable identity.

Proposition 3.13. *We have*

$$\begin{aligned}
\theta_3(z + y|\tau)\theta_3(z - y|\tau) &- \theta_3\left(z + y + \frac{\pi}{4}|\tau\right)\theta_3\left(z - y + \frac{\pi}{4}|\tau\right) \\
&+ \theta_4(z + y|\tau)\theta_4(z - y|\tau) - \theta_4\left(z + y + \frac{\pi}{4}|\tau\right)\theta_4\left(z - y + \frac{\pi}{4}|\tau\right) \\
&= 4\theta_3(2y|2\tau)\theta_2(4z|8\tau).
\end{aligned} \tag{24}$$

We show that the following identities can be deduced from (24). The first four identities can also be found in Berndt [**2**, Entry 25, p. 40].

Corollary 3.14. *We have*

(a) $\varphi(q)\psi(q^2) = \psi^2(q)$, (b) $\varphi(q) - \varphi(-q) = 4q\psi(q^8)$,

(c) $\varphi(q) + \varphi(-q) = 2\varphi(q^4)$, (d) $\varphi^2(q) - \varphi^2(-q) = 8q\psi^2(q^2)$,

(e) $\psi^2(q) - \varphi(-q)\psi(q^2) = 4q\psi(q^2)\psi(q^8)$, (f) $\psi^2(q) + \varphi(-q)\psi(q^2) = 2\psi(q^2)\varphi(q^4)$.

Proof. We only prove (a) in detail.

(a) Setting $y = 0$ and $z = \dfrac{\pi\tau}{2}$ in (24), we have

$$\theta_3^2\Big(\frac{\pi\tau}{2}|\tau\Big) - \theta_3^2\Big(\frac{\pi\tau}{2}+\frac{\pi}{4}|\tau\Big) + \theta_4^2\Big(\frac{\pi\tau}{2}|\tau\Big) - \theta_4^2\Big(\frac{\pi\tau}{2}+\frac{\pi}{4}|\tau\Big) = 4\theta_3(0|2\tau)\theta_2(2\pi\tau|8\tau).$$

Applying (6) to the above identity gives

$$\theta_2^2(0|\tau) - i\theta_2^2\Big(\frac{\pi}{4}|\tau\Big) + \theta_1^2(0|\tau) + i\theta_1^2\Big(\frac{\pi}{4}|\tau\Big) = 4q^{\frac{1}{4}}\theta_3(0|2\tau)\theta_2(2\pi\tau|8\tau).$$

Note that $\theta_1(0|\tau) = 0$ by (2) and $\theta_2{}^2\Big(\frac{\pi}{4}|\tau\Big) = \theta_1^2\Big(\frac{\pi}{4}|\tau\Big)$ by (5). Then the above identity reduces to

$$\theta_2^2(0|\tau) = 4q^{\frac{1}{4}}\theta_3(0|2\tau)\theta_2(2\pi\tau|8\tau).$$

Combining (2) and (7) with the above identity gives what we need.

(b) Take $y = \dfrac{\pi}{4}$ and $z = 0$ in (24).

(c) Take $y = \dfrac{\pi}{4}$ and $z = \pi\tau$ in (24).

(d) This identity can be proved by either combining the above three identities or taking $y = z = \dfrac{\pi\tau}{2}$ in (24).

(e) Take $y = \dfrac{\pi\tau}{2}$ and $z = 0$ in (24).

(f) Take $y = \dfrac{\pi\tau}{2}$ and $z = \pi\tau$ in (24). $\square$

Obviously, more modular identities can be deduced from (24).

4. PROOFS OF THE TWO FORMULAS FOR $(q;q)_\infty^{2n}$

In this section, we prove two formulas for $(q;q)_\infty^{2n}$ from two special cases of $H_{m,n}(y_1, y_2, \dots, y_n)$ in Theorem 1.7, respectively.

Proof of Corollary 1.10. Note that

$$\prod_{j=1}^{n}\theta_3\left(z+\frac{(2j-1)\pi}{4n}|\tau\right)\theta_3\left(z-\frac{(2j-1)\pi}{4n}|\tau\right) = \frac{(q;q)_\infty^{2n}}{(q^{2n};q^{2n})_\infty}\theta_3(2nz|2n\tau).$$

In (23), we set $y_j = \frac{(2j-1)\pi}{4n}$ and $y_{n+j} = -\frac{(2j-1)\pi}{4n}$ for $1 \leq j \leq n$. Then the left-hand side of (23) equals

$$\begin{aligned}&\sum_{k=0}^{2mn-1} \prod_{j=1}^{n} \theta_3\left(z + \frac{(2j-1)\pi}{4n} + \frac{k\pi}{2mn}|\tau\right) \theta_3\left(z - \frac{(2j-1)\pi}{4n} + \frac{k\pi}{2mn}|\tau\right)\\ &\qquad = \frac{(q;q)_\infty^{2n}}{(q^{2n};q^{2n})_\infty} \sum_{k=0}^{2mn-1} (-1)^k \theta_3\left(2nz + \frac{k\pi}{m}|2n\tau\right).\end{aligned}$$

Hence we have

$$\begin{aligned}&\sum_{k=0}^{2mn-1} (-1)^k \theta_3\left(2nz + \frac{k\pi}{m}|2n\tau\right)\\ &= \frac{(q^{2n};\ q^{2n})_\infty}{(q;\ q)_\infty^{2n}} \theta_2(2mnz|2m^2n\tau) \times \\ &\quad H_{m,2n}\left(\frac{\pi}{4n}, \frac{3\pi}{4n}, \cdots, \frac{(2n-1)\pi}{4n}, -\frac{\pi}{4n}, -\frac{3\pi}{4n}, \cdots, -\frac{(2n-1)\pi}{4n}|\tau\right).\end{aligned} \tag{25}$$

The left-hand side of the above identity is

$$\begin{aligned}\sum_{\ell=0}^{2n-1} \sum_{k=\ell m}^{(\ell+1)m-1} (-1)^k \theta_3\left(2nz + \frac{k\pi}{m}|2n\tau\right) &= \sum_{\ell=0}^{2n-1} \sum_{k=0}^{m-1} (-1)^{k+\ell m} \theta_3\left(2nz + \ell\pi + \frac{k\pi}{m}|2n\tau\right)\\ &= \sum_{\ell=0}^{2n-1} \sum_{k=0}^{m-1} (-1)^{k+\ell m} \theta_3\left(2nz + \frac{k\pi}{m}|2n\tau\right).\end{aligned}$$

Substitute the above identity back into (25) to obtain

$$\begin{aligned}&\sum_{\ell=0}^{2n-1} \sum_{k=0}^{m-1} (-1)^{k+\ell m} \theta_3\left(2nz + \frac{k\pi}{m}|2n\tau\right)\\ &= \frac{(q^{2n};\ q^{2n})_\infty}{(q;\ q)_\infty^{2n}} H_{m,2n}\left(\frac{\pi}{4n}, \frac{3\pi}{4n}, \cdots, \frac{(2n-1)\pi}{4n}, -\frac{\pi}{4n}, -\frac{3\pi}{4n}, \cdots, -\frac{(2n-1)\pi}{4n}|\tau\right)\\ &\quad \times \theta_2(2mnz|2m^2n\tau).\end{aligned}$$

Note that the left-hand side of the above identity is 0 when m is odd. When m is even, replace m by $2m$ in the above identity to get

$$\begin{aligned}&2n \sum_{k=0}^{2m-1} (-1)^k \theta_3\left(2nz + \frac{k\pi}{2m}|2n\tau\right) = \frac{(q^{2n};\ q^{2n})_\infty}{(q;\ q)_\infty^{2n}} \theta_2(4mnz|8m^2n\tau) \times \\ &\quad H_{2m,2n}\left(\frac{\pi}{4n}, \frac{3\pi}{4n}, \cdots, \frac{(2n-1)\pi}{4n}, -\frac{\pi}{4n}, -\frac{3\pi}{4n}, \cdots, -\frac{(2n-1)\pi}{4n}|\tau\right).\end{aligned}$$

Simplifying the left-hand side of the above identity by (14), we obtain

$$H_{2m,2n}\left(\frac{\pi}{4n},\frac{3\pi}{4n},\cdots,\frac{(2n-1)\pi}{4n},-\frac{\pi}{4n},-\frac{3\pi}{4n},\cdots,-\frac{(2n-1)\pi}{4n}|\tau\right)=\frac{4mn(q;\ q)_\infty^{2n}}{(q^{2n};\ q^{2n})_\infty}.$$

Then, combining with (16), we obtain

$$(q;q)_\infty^{2n}=q^{-m^2n}\ (q^{2n};q^{2n})_\infty\sum_{\substack{s_1,s_2\dots,s_{2n}=-\infty\\ s_1+s_2+\cdots+s_{2n}=2mn}}^{\infty}q^{\frac{1}{2}(s_1^2+s_2^2+\cdots+s_{2n}^2)}e^{\frac{\pi i}{2n}\sum_{l=1}^{n}(s_l-s_{n+l})(2l-1)}.$$

Note the left-hand side of the above identity is independent of m. Setting $m=1$, we get (19), which ends the proof. □

Proof of Corollary 1.11. We put $y_j=\dfrac{(2j-1)\pi\tau}{4n}$ and $y_{n+j}=-\dfrac{(2j-1)\pi\tau}{4n}$ in (23), for $1\le j\le n$. Note the fact that

$$\prod_{j=1}^{n}\theta_3\left(x+\frac{(2j-1)\pi\tau}{4n}|\tau\right)\theta_3\left(x-\frac{(2j-1)\pi\tau}{4n}|\tau\right)=\frac{(q;q)_\infty^{2n}}{\left(q^{\frac{1}{2n}};q^{\frac{1}{2n}}\right)_\infty}\theta_3\left(x|\frac{\tau}{2n}\right).$$

In a similar way, we obtain

$$(q^{2n};q^{2n})_\infty^{2n}=q^{-\frac{m^2n}{2}}(q;q)_\infty\sum_{\substack{s_1,s_2\dots,s_{2n}=-\infty\\ s_1+s_2+\cdots+s_{2n}=mn}}^{\infty}q^{n(s_1^2+s_2^2+\cdots+s_{2n}^2)+\frac{1}{4}\sum_{l=1}^{n}(s_l-s_{n+l})(2l-1)}.$$

Setting $m=1$, we get (20). This completes the proof. □

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