# ON THE COEFFICIENTS OF AN ASYMPTOTIC EXPANSION RELATED TO SOMOS' QUADRATIC RECURRENCE CONSTANT

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In this paper we study the coefficients of an asymptotic expansion related to Somos' Quadratic Recurrence Constant. We develop recurrence relations and an asymptotic estimation for them. We also show that these coefficients are related to the Ordered Bell Numbers.

## 1. INTRODUCTION

Somos' Quadratic Recurrence Constant, named after M. SOMOS [**2**, p. 446] [**5**], is the number

$$\sigma = \sqrt{1\sqrt{2\sqrt{3\sqrt{4\cdots}}}} = 1.6616879496\ldots$$

The constant $\sigma$ arose when he had examined the asymptotic behavior of the sequence

$$(1) \qquad g_0 = 1, \ g_n = ng_{n-1}^2, \ n \geq 1,$$

with first few terms $1, 1, 2, 12, 576, 1658880, \ldots$ SOMOS showed that $g_n$ has an asymptotic series as follows:

$$(2) \qquad \frac{\sigma^{2^n}}{ng_n} \sim a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + \frac{a_3}{n^3} + \frac{a_4}{n^4} + \cdots, \ n \to +\infty,$$

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where the first few coefficients $a_k$ are $1, 2, -1, 4, -21, 138, \ldots$ (this is sequence A116603 in the On-line Encyclopedia of Integer Sequences). From (1) it can be shown that the generating function

$$A(x) = \sum_{k \geq 0} a_k x^k$$

satisfies the functional equation

(3) $$A^2(x) = (1 + x)^2 A \left( \frac{x}{1 + x} \right).$$

J. SONDOW and P. HADJICOSTAS [**6**] showed that for $x \geq 0$

(4) $$A(x) = \prod_{k \geq 1} (1 + kx)^{1/2^k}.$$

The purpose of this paper is to develop recurrence relations and an asymptotic approximation for the $a_k$'s.

## 2. MAIN RESULTS

Our first theorem gives a recurrence relation for the coefficients $a_k$ in the asymptotic expansion (2).

**Theorem 1.** *The coefficients $a_k$ satisfy the recurrence relation for $k \geq 3$.*

$$a_0 = 1, \; a_1 = 2, \; a_2 = -1, \; a_k = \sum_{j=1}^{k-1} \left( (-1)^{k-j} \binom{k-3}{k-j} a_j - a_{k-j} a_j \right).$$

In the second theorem we show that the generating function $A(x)$ can be expressed in terms of the Ordered Bell Numbers [**7**, p. 189], *i.e.*, the number of ordered partitions of the set $\{1, \ldots, n\}$. The Ordered Bell Numbers $b_k$ have the following exponential generating function

$$\frac{1}{2 - e^x} = \sum_{k \geq 0} \frac{b_k}{k!} x^k.$$

They are given explicitly by the formula

$$b_k = \sum_{j \geq 0} \frac{j^k}{2^{j+1}}.$$

From (4) we will prove

**Theorem 2.** *The generating function of the coefficients $a_k$ has the following representation*

$$(5) \qquad A(x) = \exp\left(\sum_{k\geq 1} \frac{(-1)^{k-1} 2b_k}{k} x^k\right).$$

Note that this is only a formal expression due to the rapid growth of the $b_k$'s (see (10)). As a corollary we represent the generating function of the Ordered Bell Numbers in terms of the logarithmic derivative of $A(x)$.

**Corollary 1.** *The generating function $B(x)$ of the Ordered Bell Numbers is given by*

$$B(x) = 1 + \frac{x}{2} \frac{A'(-x)}{A(-x)} = \sum_{k\geq 0} b_k x^k.$$

Theorem 2 allows us to obtain a recurrence relation for the coefficients $a_k$ using the Ordered Bell Numbers.

**Theorem 3.** *For every $k \geq 1$ we have*

$$(6) \qquad a_0 = 1, \; a_k = \frac{1}{k} \sum_{j=1}^{k} (-1)^{j-1} 2b_j a_{k-j}.$$

Finally, from a theorem of E. A. BENDER [**1**, Theorem 2] (quoted in [**4**, Theorem 7.3]), we give an asymptotic estimation for the $a_k$'s.

**Theorem 4.** *We have the following asymptotic approximation*

$$(7) \qquad a_k = (-1)^{k-1} \frac{(k-1)!}{\log^{k+1} 2} \left(1 + \mathcal{O}\left(\frac{1}{k}\right)\right)$$

*as $k \to +\infty$.*

The table below shows the exact and the approximate values (rounded to the nearest integer) given by (7) of the coefficients $a_k$ for some $k$. Relative errors are displayed as well.

| $k$ | Exact | Approximation | Relative error |
|---|---|---|---|
| 1 | 2 | 2 | 0.00 |
| 2 | $-1$ | $-3$ | 2.00 |
| 5 | 138 | 216 | 0.57 |
| 10 | $-17070418$ | $-20449513$ | 0.20 |
| 15 | 27552372478592 | 30704425463714 | 0.11 |
| 20 | $-247772400254700937149$ | $-267768779624438420309$ | 0.08 |

Table 1. The exact and the estimated values of the coefficients $a_k$ for some values of $k$.

## 3. THE PROOFS OF THE THEOREMS

**Proof of Theorem 1.** A formal manipulation gives

$$A\left(\frac{x}{1+x}\right) = \sum_{k \geq 0} a_k \left(\frac{x}{1+x}\right)^k = \sum_{k \geq 0} a_k x^k \sum_{j \geq 0} \binom{-k}{j} x^j$$

$$= \sum_{k \geq 0} \left( \sum_{j=0}^{k} \binom{-j}{k-j} a_j \right) x^k = \sum_{k \geq 0} \left( \sum_{j=0}^{k} (-1)^{k-j} \binom{k-1}{k-j} a_j \right) x^k.$$

Let

$$A_{k,j} := (-1)^{k-j} \binom{k-1}{k-j} a_j,$$

then we find

$$(1+x)^2 A\left(\frac{x}{1+x}\right) - a_0 - (2a_0 + a_1) x =$$

$$= \sum_{k \geq 2} \left( \sum_{j=0}^{k} A_{k,j} + 2 \sum_{j=0}^{k-1} A_{k-1,j} + \sum_{j=0}^{k-2} A_{k-2,j} \right) x^k$$

$$= \sum_{k \geq 2} \left( a_k - (k-3) a_{k-1} + \sum_{j=0}^{k-2} (A_{k,j} + 2A_{k-1,j} + A_{k-2,j}) \right) x^k$$

$$= \sum_{k \geq 2} \left( a_k - (k-3) a_{k-1} + \sum_{j=0}^{k-2} (-1)^{k-j} \binom{k-3}{k-j} a_j \right) x^k$$

$$= \sum_{k \geq 2} \left( \sum_{j=0}^{k} (-1)^{k-j} \binom{k-3}{k-j} a_j \right) x^k.$$

Hence

$$(8) \qquad (1+x)^2 A\left(\frac{x}{1+x}\right) = \sum_{k \geq 0} \left( \sum_{j=0}^{k} (-1)^{k-j} \binom{k-3}{k-j} a_j \right) x^k.$$

On the other hand we have

$$(9) \qquad A^2(x) = \left( \sum_{k \geq 0} a_k x^k \right)^2 = \sum_{k \geq 0} \left( \sum_{j=0}^{k} a_{k-j} a_j \right) x^k.$$

Thus from (8) and (9) by (3) it follows that

$$\sum_{j=0}^{k} a_{k-j} a_j = \sum_{j=0}^{k} (-1)^{k-j} \binom{k-3}{k-j} a_j,$$

and hence

$$a_k = \sum_{j=1}^{k-1} \left( (-1)^{k-j} \binom{k-3}{k-j} a_j - a_{k-j} a_j \right),$$

for $k \geq 3$.                                                                                   □

**Proof of Theorem 2.** Simple formal manipulation gives

$$\log A(x) = \log \left( \prod_{j \geq 1} (1+jx)^{1/2^j} \right) = \sum_{j \geq 1} \frac{1}{2^j} \log (1+jx)$$

$$= \sum_{j \geq 1} \frac{1}{2^j} \sum_{k \geq 1} \frac{(-1)^{k-1} j^k}{k} x^k = \sum_{k \geq 1} \left( \sum_{j \geq 1} \frac{j^k}{2^{j+1}} \right) \frac{(-1)^{k-1} 2}{k} x^k$$

$$= \sum_{k \geq 1} \frac{(-1)^{k-1} 2b_k}{k} x^k.$$                                                   □

**Proof of Corollary 1.** From (5) we find

$$1 + \frac{x}{2} \frac{A'(-x)}{A(-x)} = 1 + \frac{x}{2} \sum_{k \geq 1} \frac{(-1)^{k-1} 2b_k}{k} k (-x)^{k-1} = 1 + \sum_{k \geq 1} b_k x^k = \sum_{k \geq 0} b_k x^k.$$   □

**Proof of Theorem 3.** From (5) we have

$$\exp \left( \sum_{k \geq 1} \frac{(-1)^{k-1} 2b_k}{k} x^k \right) = \sum_{k \geq 0} a_k x^k.$$

By differentiating each side with respect to $x$, we obtain

$$\left( \sum_{k \geq 1} (-1)^{k-1} 2b_k x^{k-1} \right) \exp \left( \sum_{k \geq 1} \frac{(-1)^{k-1} 2b_k}{k} x^k \right) = \sum_{k \geq 0} k a_k x^{k-1},$$

$$\left( \sum_{k \geq 0} (-1)^k 2b_{k+1} x^k \right) \left( \sum_{k \geq 0} a_k x^k \right) = \sum_{k \geq 0} (k+1) a_{k+1} x^k,$$

$$\sum_{k \geq 0} \left( \sum_{j=0}^{k} (-1)^j 2b_{j+1} a_{k-j} \right) x^k = \sum_{k \geq 0} (k+1) a_{k+1} x^k.$$

Equating the coefficients in both sides gives (6).                                                □

**Theorem 5** (E. A. BENDER). *Suppose that*

$$\alpha(x) = \sum_{k \geq 1} \alpha_k x^k, \quad F(x,y) = \sum_{h,k \geq 0} f_{hk} x^h y^k,$$

$$\beta(x) = \sum_{k \geq 0} \beta_k x^k = F(x, \alpha(x)), \quad D(x) = \sum_{k \geq 0} \delta_k x^k = \left. \frac{\partial F(x,y)}{\partial y} \right|_{y=\alpha(x)}.$$

*Assume that $F(x, y)$ is analytic in $x$ and $y$ in a neighborhood of $(0,0)$, $\alpha_k \neq 0$ and*

1. $$\alpha_{k-1} = o(\alpha_k) \ \text{ as } k \to +\infty,$$

2. $$\sum_{j=r}^{k-r} |\alpha_j \alpha_{k-j}| = \mathcal{O}(\alpha_{k-r}) \ \text{ for some } r > 0 \text{ as } k \to +\infty.$$

*Then $\beta_k = \sum_{j=0}^{r-1} \delta_j \alpha_{k-j} + \mathcal{O}(\alpha_{k-r})$ as $k \to +\infty$.*

**Proof of Theorem 4.** We apply Theorem 5 to the functions

$$\alpha(x) := \sum_{k \geq 1} \frac{(-1)^{k-1} 2b_k}{k} x^k, \quad F(x, y) := e^y.$$

It follows that $\beta(x) = D(x) = A(x) = \sum_{k \geq 0} a_k x^k$.

H. S. WILF [**7**, p. 190] showed that

(10) $$b_k = \frac{k!}{2 \log^{k+1} 2} \left(1 + \mathcal{O}((0.16 \log 2)^k)\right) = \frac{k!}{2 \log^{k+1} 2} \left(1 + \mathcal{O}(0.12^k)\right)$$

as $k \to +\infty$ (a complete asymptotic expansion can be found in [**3**, p. 269]). Hence

$$\alpha_k := \frac{(-1)^{k-1} 2b_k}{k} = (-1)^{k-1} \frac{(k-1)!}{\log^{k+1} 2} \left(1 + \mathcal{O}(0.12^k)\right)$$

and

(11) $$c_1 \frac{(k-1)!}{\log^{k+1} 2} < |\alpha_k| < c_2 \frac{(k-1)!}{\log^{k+1} 2}$$

for some $c_2 > c_1 > 0$. Since $\alpha_k \neq 0$ and

$$\lim_{k \to +\infty} \frac{\alpha_{k-1}}{\alpha_k} = \lim_{k \to +\infty} \frac{(-1)^k \frac{(k-2)!}{\log^k 2} \left(1 + \mathcal{O}(0.12^{k-1})\right)}{(-1)^{k-1} \frac{(k-1)!}{\log^{k+1} 2} \left(1 + \mathcal{O}(0.12^k)\right)}$$

$$= -\lim_{k \to +\infty} \frac{\log 2}{k-1} \left(1 + \mathcal{O}(0.12^{k-1})\right) = 0,$$

the condition (2) of Theorem 5 holds. From (11)

$$\sum_{j=1}^{k-1} |\alpha_j \alpha_{k-j}|$$

$$< c_2^2 \sum_{j=1}^{k-1} \frac{(j-1)!}{\log^{j+1} 2} \frac{(k-j-1)!}{\log^{k-j+1} 2} = \frac{c_2^2}{\log^2 2} \frac{(k-2)!}{\log^k 2} \sum_{j=1}^{k-1} \frac{(j-1)!\,(k-j-1)!}{(k-2)!}$$

$$= \frac{c_2^2}{\log^2 2} \frac{(k-2)!}{\log^k 2} \sum_{j=0}^{k-2} \frac{1}{\binom{k-2}{j}} = \mathcal{O}\left( \frac{(k-2)!}{\log^k 2} \right) = \mathcal{O}\left( \alpha_{k-1} \right).$$

Hence condition (2) holds with $r = 1$. Since $F(x,y) = e^y$ is analytic in $x$ and $y$, it follows that

$$a_k = a_0 \alpha_k + \mathcal{O}(\alpha_{k-1}) = (-1)^{k-1} \frac{(k-1)!}{\log^{k+1} 2} \left( 1 + \mathcal{O}(0.12^k) \right) + \mathcal{O}\left( \frac{(k-2)!}{\log^k 2} \right)$$

$$= (-1)^{k-1} \frac{(k-1)!}{\log^{k+1} 2} \left( 1 + \mathcal{O}\left( \frac{1}{k} \right) \right)$$

as $k \to +\infty$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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